

# ON FUSION CATEGORIES WITH FEW IRREDUCIBLE DEGREES

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**ABSTRACT.** We prove some results on the structure of certain classes of integral fusion categories and semisimple Hopf algebras under restrictions on the set of its irreducible degrees.

## 1. INTRODUCTION

Let  $k$  be an algebraically closed field of characteristic zero. Let  $\mathcal{C}$  be a fusion category over  $k$ . That is,  $\mathcal{C}$  is a  $k$ -linear semisimple rigid tensor category with finitely many isomorphism classes of simple objects, finite-dimensional spaces of morphisms, and such that the unit object  $\mathbf{1}$  of  $\mathcal{C}$  is simple.

For example, if  $G$  is a finite group, then the categories  $\text{Rep } G$  of its finite dimensional representations, and the category  $\mathcal{C}(G, \omega)$  of  $G$ -graded vector spaces with associativity determined by the 3-cocycle  $\omega$ , are fusion categories over  $k$ . More generally, if  $H$  is a finite dimensional semisimple quasi-Hopf algebra over  $k$ , then the category  $\text{Rep } H$  of its finite dimensional representations is a fusion category.

Let  $\text{Irr}(\mathcal{C})$  denote the set of isomorphism classes of simple objects in the fusion category  $\mathcal{C}$ . In analogy with the case of finite groups [12], we shall use the notation  $\text{c.d.}(\mathcal{C})$  to indicate the set

$$\text{c.d.}(\mathcal{C}) = \{\text{FPdim } x \mid x \in \text{Irr}(\mathcal{C})\}.$$

Here,  $\text{FPdim } x$  denotes the *Frobenius-Perron dimension* of  $x \in \text{Irr}(\mathcal{C})$ . Notice that, when  $\mathcal{C}$  is the representation category of a quasi-Hopf algebra, Frobenius-Perron dimensions coincide with the dimension of the underlying vector spaces. In this case, we shall use the notation  $\text{c.d.}(\mathcal{C}) = \text{c.d.}(H)$ .

The positive real numbers  $\text{FPdim } x$ ,  $x \in \text{Irr}(\mathcal{C})$ , will be called the *irreducible degrees* of  $\mathcal{C}$ .

The fusion categories that we shall consider in this paper are all *integral*, that is, the Frobenius-Perron dimensions of objects of  $\mathcal{C}$  are (natural) integers. By [7, Theorem 8.33],  $\mathcal{C}$  is isomorphic to the category of representations of some finite dimensional semisimple quasi-Hopf algebra.

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For a finite group  $G$ , the knowledge of the set  $\text{c.d.}(G) = \text{c.d.}(kG)$  gives in some cases substantial information about the structure of  $G$ . It is known, for instance, that if  $|\text{c.d.}(G)| \leq 3$ , then  $G$  is solvable.

On the other hand, if  $|\text{c.d.}(G)| = 2$ , say  $\text{c.d.}(G) = \{1, m\}$ ,  $m \geq 1$ , then either  $G$  has an abelian normal subgroup of index  $m$  or else  $G$  is nilpotent of class  $\leq 3$ . Furthermore, if  $G$  is nonabelian, then  $\text{c.d.}(G) = \{1, p\}$  for some prime number  $p$ , if and only if  $G$  contains an abelian normal subgroup of index  $p$  or the center  $Z(G)$  has index  $p^3$ . See [12, Theorems (12.11), (12.14) and (12.15)].

In the context of semisimple Hopf algebras, some results in the same spirit are known. A basic one is that of [37], which asserts that if  $|\text{c.d.}(H)| \leq 3$ , then  $G(H^*)$  is not trivial, in other words,  $H$  has nontrivial characters of degree 1. A similar result appears in [18, Theorem 2.2.3].

Further results, leading to classification theorems in some specific cases, appear in the work of Izumi and Kosaki [13] for Kac algebras, that is, Hopf  $C^*$ -algebras.

In this paper we consider the general problem of understanding the structure of a fusion category  $\mathcal{C}$  after the knowledge of  $\text{c.d.}(\mathcal{C})$ . For instance, it is well-known that  $\text{c.d.}(\mathcal{C}) = \{1\}$  if and only if  $\mathcal{C}$  is pointed, if and only if  $\mathcal{C} \simeq \mathcal{C}(G, \omega)$ , for some 3-cocycle  $\omega$  on the group  $G = G(\mathcal{C})$  of isomorphism classes of invertible objects of  $\mathcal{C}$ .

More specifically, we address the following question:

**Question 1.1.** *Suppose  $\text{c.d.}(\mathcal{C}) = \{1, p\}$ , with  $p$  a prime number. What can be said about the structure of  $\mathcal{C}$ ?*

We treat mostly structural questions regarding nilpotency and solvability, in the sense introduced in [11] and [8]. (A related question for semisimple Hopf algebras, that we shall not discuss in the present paper, was posed in [24, Question 7.2].)

The notions of nilpotency and solvability of a fusion category are related to the corresponding notions for finite groups as follows: if  $G$  is a finite group, then the category  $\text{Rep } G$  is nilpotent or solvable if and only if  $G$  is nilpotent or solvable, respectively. On the dual side, a pointed fusion category  $\mathcal{C}(G, \omega)$  is always nilpotent, while it is solvable if and only if the group  $G$  is solvable.

An important class of fusion categories, called *weakly group-theoretical* fusion categories, was introduced and studied in [8]. This generalized in turn the notion of a group-theoretical fusion category of [7]. Roughly,  $\mathcal{C}$  is group-theoretical if it is Morita equivalent to a pointed fusion category, and it is weakly group-theoretical if it is obtained from the trivial fusion category  $\text{Vec}_k$ , of finite dimensional vector spaces, by means of successive group extensions and equivariantizations. Every nilpotent or solvable fusion category is weakly group-theoretical.

With regard to Question 1.1, consider for instance the case where  $\mathcal{C} = \text{Rep } H$ , for a semisimple Hopf algebra  $H$ . A result in this direction is known in the case  $p = 2$ . It is shown in [2, Corollary 6.6] that if  $H$  is a semisimple Hopf algebra such that  $\text{c.d.}(H) \subseteq \{1, 2\}$ , then  $H$  is upper semisolvable. Moreover,  $H$  is necessarily cocommutative if  $G(H^*)$  is of order 2. The proof of these results relies on a refinement of a theorem of Nichols and Richmond ([25, Theorem 11]) given in [2, Theorem 1.1].

In the context of Kac algebras, it is shown in [13, Theorem IX.8 (iii)] that if  $\text{c.d.}(H^*) = \{1, p\}$  and in addition  $|G(H)| = p$ , then  $H$  is a central abelian extension associated to an action of the cyclic group of order  $p$  on a nilpotent group. In the recent terminology introduced in [11], this result implies that such a Kac algebra is *nilpotent*. See Remark 4.5.

The main results of this paper are summarized in the following theorem.

**Theorem 1.2.** *Let  $\mathcal{C}$  be a fusion category over  $k$ . Then we have:*

(i) *Suppose  $\mathcal{C}$  is weakly group-theoretical and has odd-dimension. Then  $\mathcal{C}$  is solvable. (Proposition 7.1.)*

*Let  $p$  be a prime number.*

(ii) *Suppose  $\text{c.d.}(\mathcal{C}) \subseteq \{1, p\}$ . Then  $\mathcal{C}$  is solvable in any of the following cases:*

- $\mathcal{C}$  is braided and odd-dimensional. (Theorem 7.3.)
- $\mathcal{C} = \mathcal{C}(G, \omega, \mathbb{Z}_p, \alpha)$  (a group-theoretical fusion category [7]) and  $G(\mathcal{C})$  is of order  $p$ . (Corollary 5.4.)
- $\mathcal{C}$  is a near-group category [33]. (Theorem 6.2.)
- $\mathcal{C} = \text{Rep } H$ , where  $H$  is a semisimple quasitriangular Hopf algebra and  $p = 2$ . (Theorem 6.12.)

(iii) *Let  $H$  be a semisimple Hopf algebra such that  $\text{c.d.}(H) \subseteq \{1, p\}$ . Then  $H^*$  is nilpotent in any of the following cases:*

- $|G(H^*)| = p$  and  $p$  divides  $|G(H)|$ . (Proposition 4.8.)
- $|G(H^*)| = p$  and  $H$  is quasitriangular. (Proposition 4.9.)
- $H$  is of type  $(1, p; p, 1)$  as an algebra. (Proposition 4.12.)

(iv) *Let  $H$  be a semisimple Hopf algebra such that  $\text{c.d.}(H) \subseteq \{1, 2\}$ . Then we have:*

- $H$  is weakly group-theoretical, and furthermore, it is group-theoretical if  $H = H_{\text{ad}}$ . (Theorem 6.4.)
- The group  $G(H)$  is solvable. (Corollary 6.9.)

(v) *Let  $H$  be a semisimple Hopf algebra of type  $(1, p; p, 1)$  as an algebra. Then  $H$  is a twisting deformation of a group algebra  $kN$ , where  $N$  is a solvable group.*

The proof of part (i) of the theorem is a consequence of the Feit-Thompson Theorem [9] that asserts that every finite group of odd order is solvable.

By [24, Corollary 4.5], the semisimple Hopf algebras  $H$  in part (iii) of the theorem are *lower semisolvable* in the sense of [17].

The results on semisimple Hopf algebras  $H$  with  $\text{c.d.}(H) \subseteq \{1, 2\}$  rely on the results of the paper [2]. We also make strong use of several results of the papers [8], [11] and [10] on weakly group-theoretical, solvable and nilpotent fusion categories.

The paper is organized as follows. In Section 2 we recall the main notions and results relevant to the problem we consider. In particular, several properties of group-theoretical fusion categories and Hopf algebra extensions are discussed here. The results on nilpotency are contained in Sections 3 and 4. The strategy in these sections consists in reducing the problem to considering Hopf algebra extensions. Sections 5, 6 and 7 are devoted to the solvability question in different situations.

## 2. PRELIMINARIES

**2.1. Fusion categories.** A *fusion category* over  $k$  is a  $k$ -linear semisimple rigid tensor category  $\mathcal{C}$  with finitely many isomorphism classes of simple objects, finite-dimensional spaces of morphisms, and such that the unit object  $\mathbf{1}$  of  $\mathcal{C}$  is simple. We refer the reader to [1, 7] for basic definitions and facts concerning fusion categories. In particular, if  $H$  is a semisimple (quasi-)Hopf algebra over  $k$ , then  $\text{Rep } H$  is a fusion category.

A *fusion subcategory* of a fusion category  $\mathcal{C}$  is a full tensor subcategory  $\mathcal{C}' \subseteq \mathcal{C}$  such that if  $X \in \mathcal{C}$  is isomorphic to a direct summand of an object of  $\mathcal{C}'$ , then  $X \in \mathcal{C}'$ . A fusion subcategory is necessarily rigid, so it is indeed a fusion category [5, Corollary F.7 (i)].

A *pointed fusion category* is a fusion category where all simple objects are invertible. A pointed fusion category is equivalent to the category  $\mathcal{C}(G, \omega)$ , of finite-dimensional  $G$ -graded vector spaces with associativity constraint determined by a cohomology class  $\omega \in H^3(G, k^\times)$ , for some finite group  $G$ . In other words,  $\mathcal{C}(G, \omega)$  is the category of representations of the quasi-Hopf algebra  $k^G$ , with associator  $\omega \in (k^G)^{\otimes 3}$ .

The fusion subcategory *generated* by a collection  $\mathcal{X}$  of objects of  $\mathcal{C}$  is the smallest fusion subcategory containing  $\mathcal{X}$ .

If  $\mathcal{C}$  is a fusion category, then the set of isomorphism classes of invertible objects of  $\mathcal{C}$  forms a group, denoted  $G(\mathcal{C})$ . The fusion subcategory generated by the invertible objects of  $\mathcal{C}$  is a fusion subcategory, denoted  $\mathcal{C}_{pt}$ ; it is the maximal pointed subcategory of  $\mathcal{C}$ .

Let  $\text{Irr}(\mathcal{C})$  denote the set of isomorphism classes of simple objects in the fusion category  $\mathcal{C}$ . The set  $\text{Irr}(\mathcal{C})$  is a basis over  $\mathbb{Z}$  of the Grothendieck ring  $\mathcal{G}(\mathcal{C})$ .

**2.2. Irreducible degrees.** For  $x \in \text{Irr}(\mathcal{C})$ , let  $\text{FPdim } x$  be its Frobenius-Perron dimension. The positive real numbers  $\text{FPdim } x$ ,  $x \in \text{Irr}(\mathcal{C})$  will be called the *irreducible degrees* of  $\mathcal{C}$ . These extend to a ring homomorphism  $\text{FPdim} : \mathcal{G}(\mathcal{C}) \rightarrow \mathbb{R}$ . When  $\mathcal{C}$  is the representation category of a quasi-Hopf

algebra, Frobenius-Perron dimensions coincide with the dimension of the underlying vector spaces.

The set of *irreducible degrees* of  $\mathcal{C}$  is defined as

$$\text{c.d.}(\mathcal{C}) = \{\text{FPdim } x \mid x \in \text{Irr}(\mathcal{C})\}.$$

The category  $\mathcal{C}$  is called *integral* if  $\text{c.d.}(\mathcal{C}) \subseteq \mathbb{N}$ .

If  $X$  is any object of  $\mathcal{C}$ , then its class  $x$  in  $\mathcal{G}(\mathcal{C})$  decomposes as  $x = \sum_{y \in \text{Irr}(\mathcal{C})} m(y, x)y$ , where  $m(y, x) = \dim \text{Hom}(Y, X)$  is the multiplicity of  $Y$  in  $X$ , if  $Y$  is an object representing the class  $y \in \text{Irr}(\mathcal{C})$ .

For all objects  $X, Y, Z \in \mathcal{C}$ , we have:

$$(2.1) \quad m(X, Y \otimes Z) = m(Y^*, Z \otimes X^*) = m(Y, X \otimes Z^*).$$

Let  $x \in \text{Irr}(\mathcal{C})$ . The stabilizer of  $x$  under left multiplication by elements of  $G(\mathcal{C})$  in the Grothendieck ring will be denoted by  $G[x]$ . So that an invertible element  $g \in G(\mathcal{C})$  belongs to  $G[x]$  if and only if  $gx = x$ .

In view of (2.1), for all  $x \in \text{Irr}(\mathcal{C})$ , we have

$$G[x] = \{g \in G(\mathcal{C}) : m(g, xx^*) > 0\} = \{g \in G(\mathcal{C}) : m(g, xx^*) = 1\}.$$

In particular, we have the following relation in  $\mathcal{G}(\mathcal{C})$ :

$$xx^* = \sum_{g \in G[x]} g + \sum_{\text{FPdim } y > 1} m(y, xx^*)y.$$

*Remark 2.1.* Notice that an object  $g \in \mathcal{C}$  is invertible if and only if  $\text{FPdim } g = 1$ .

Suppose that  $\mathcal{C}$  is an integral fusion category with  $|\text{c.d.}(\mathcal{C})| = 2$ . That is,  $\text{c.d.}(\mathcal{C}) = \{1, d\}$  for some integer  $d > 1$ . Then  $d$  divides the order of  $G[x]$  for all  $x \in \text{Irr}(\mathcal{C})$  with  $\text{FPdim } x > 1$ . In particular,  $d$  divides the order of  $G(\mathcal{C})$ , and thus  $G(\mathcal{C}) \neq 1$ .

Indeed, if  $x \in \text{Irr}(\mathcal{C})$  with  $\text{FPdim } x = d$ , we have the relation:

$$xx^* = \sum_{g \in G[x]} g + \sum_{\text{FPdim } y = d} m(y, xx^*)y.$$

The claim follows by taking Frobenius-Perron dimensions.

**2.3. Semisimple Hopf algebras.** Let  $H$  be a semisimple Hopf algebra over  $k$ . We next recall some of the terminology and conventions from [23] that will be used throughout this paper.

As an algebra,  $H$  is isomorphic to a direct sum of full matrix algebras

$$(2.2) \quad H \simeq k^{(n)} \oplus \bigoplus_{i=1}^r M_{d_i}(k)^{(n_i)},$$

where  $n = |G(H^*)|$ . The Nichols-Zoeller theorem [26] implies that  $n$  divides both  $\dim H$  and  $n_i d_i^2$ , for all  $i = 1, \dots, r$ .

If we have an isomorphism as in (2.2), we shall say that  $H$  is of *type*  $(1, n; d_1, n_1; \dots; d_r, n_r)$  *as an algebra*. If  $H^*$  is of type  $(1, n; d_1, n_1; \dots; d_r, n_r)$  as an algebra, we shall say that  $H$  is of *type*  $(1, n; d_1, n_1; \dots; d_r, n_r)$  *as a coalgebra*.

Let  $V$  be an  $H$ -module. The *character* of  $V$  is the element  $\chi = \chi_V \in H^*$  defined by  $\chi(h) = \text{Tr}_V(h)$ , for all  $h \in H$ . For a character  $\chi$ , its *degree* is the integer  $\deg \chi = \chi(1) = \dim V$ . The character  $\chi_V$  is called irreducible if  $V$  is irreducible.

The set  $\text{Irr}(H)$  of irreducible characters of  $H$  spans a semisimple subalgebra  $R(H)$  of  $H^*$ , called the character algebra of  $H$ . It is isomorphic, under the map  $V \rightarrow \chi_V$ , to the extension of scalars  $k \otimes_{\mathbb{Z}} \mathcal{G}(\text{Rep } H)$  of the Grothendieck ring of the category  $\text{Rep } H$ . In particular, there is an identification  $\text{Irr}(H) \simeq \text{Irr}(\text{Rep } H)$ .

Under this identification, all properties listed in Subsection 2.2 hold true for characters.

In this context, we have  $G(\text{Rep } H) = G(H^*)$ . The stabilizer of  $\chi$  under left multiplication by elements in  $G(H^*)$  will be denoted by  $G[\chi]$ . By the Nichols-Zoeller theorem [26], we have that  $|G[\chi]|$  divides  $(\deg \chi)^2$ .

Following [12, Chapter 12], we shall denote  $\text{c.d.}(H) = \text{c.d.}(\text{Rep } H)$ . So that,

$$\text{c.d.}(H) = \{\deg \chi \mid \chi \in \text{Irr}(H)\}.$$

In particular, if  $H$  is of type  $(1, n; d_1, n_1; \dots; d_r, n_r)$  as an algebra, then  $\text{c.d.}(H) = \{1, d_1, \dots, d_r\}$ .

There is a bijective correspondence between Hopf algebra quotients of  $H$  and standard subalgebras of  $R(H)$ , that is, subalgebras spanned by irreducible characters of  $H$ . This correspondence assigns to the Hopf algebra quotient  $H \rightarrow \overline{H}$  its character algebra  $R(\overline{H}) \subseteq R(H)$ . See [25].

**2.4. Group-theoretical categories.** A group-theoretical fusion category is a fusion category Morita equivalent to a pointed fusion category  $\mathcal{C}(G, \omega)$ . Such a fusion category is equivalent to the category  $\mathcal{C}(G, \omega, F, \alpha)$  of  $k_\alpha F$ -bimodules in  $\mathcal{C}(G, \omega)$ , where  $G$  is a finite group,  $\omega$  is a 3-cocycle on  $G$ ,  $F \subseteq G$  is a subgroup and  $\alpha \in C^2(F, k^\times)$  is a 2-cochain on  $F$  such that  $\omega|_F = d\alpha$ . A semisimple Hopf algebra  $H$  is called group-theoretical if the category  $\text{Rep } H$  is group-theoretical.

Let  $\mathcal{C} = \mathcal{C}(G, \omega, F, \alpha)$  be a group-theoretical fusion category. Let also  $\Gamma$  be a subgroup of  $G$ , endowed with a 2-cocycle  $\beta \in Z^2(\Gamma, k^\times)$ , such that

- The class  $\omega|_\Gamma$  is trivial;
- $G = F\Gamma$ ;
- The class  $\alpha|_{F\cap\Gamma}\beta^{-1}|_{F\cap\Gamma}$  is non-degenerate.

Then there is an associated semisimple Hopf algebra  $H$ , such that the category  $\text{Rep } H$  is equivalent to  $\mathcal{C}$ . By [27], equivalence classes subgroups  $\Gamma$  of

$G$  satisfying the conditions above, classify fiber functors  $\mathcal{C} \mapsto \text{Vec}_k$ ; these correspond to the distinct Hopf algebras  $H$ .

Let  $\mathcal{C} = \mathcal{C}(G, \omega, F, \alpha)$  be a group-theoretical fusion category. The simple objects of  $\mathcal{C}$  are classified by pairs  $(s, U_s)$ , where  $s$  runs over a set of representatives of the double cosets of  $F$  in  $G$ , that is, orbits of the action of  $F$  in the space  $F \backslash G$  of left cosets of  $F$  in  $G$ ,  $F_s = F \cap sFs^{-1}$  is the stabilizer of  $s \in F \backslash G$ , and  $U_s$  is an irreducible representation of the twisted group algebra  $k_{\sigma_s} F_s$ , that is, an irreducible projective representation of  $F_s$  with respect to certain 2-cocycle  $\sigma_s$  determined by  $\omega$ . See [10, Theorem 5.1].

The irreducible representation  $W_{(s, U_s)}$  corresponding to such a pair  $(s, U_s)$  has dimension

$$(2.3) \quad \dim W_{(s, U_s)} = [F : F_s] \dim U_s.$$

As a consequence, we have:

**Corollary 2.2.** *The irreducible degrees of  $\mathcal{C}(G, \omega, F, \alpha)$  divide the order of  $F$ .*

*Remark 2.3.* A group-theoretical category  $\mathcal{C} = \mathcal{C}(G, \omega, F, \alpha)$  is an integral fusion category. An explicit construction of a quasi-Hopf algebra  $H$  such that  $\text{Rep } H \simeq \mathcal{C}$  was given in [20].

As an algebra,  $H$  is a crossed product  $k^{F \backslash G} \#_{\sigma} kF$ , where  $F \backslash G$  is the space of left cosets of  $F$  in  $G$  with the natural action of  $F$ , and  $\sigma$  is a certain 2-cocycle determined by  $\omega$ .

Irreducible representations of  $H$ , that is, simple objects of  $\mathcal{C}$ , can therefore be described using the results for group crossed products in [17]: this is done in [20, Proposition 5.5].

By [10, Theorem 5.2], the group  $G(\mathcal{C})$  of invertible objects of  $\mathcal{C}$  fits into an exact sequence

$$(2.4) \quad 1 \rightarrow \widehat{F} \rightarrow G(\mathcal{C}) \rightarrow K \rightarrow 1,$$

where  $K = \{x \in N_G(F) : [\sigma_x] = 1\}$ .

**2.5. Abelian extensions.** Suppose that  $G = F\Gamma$  is an exact factorization of the finite group  $G$ , where  $\Gamma$  and  $F$  are subgroups of  $G$ . Equivalently,  $F$  and  $\Gamma$  form a *matched pair* of groups with the actions  $\triangleleft: \Gamma \times F \rightarrow \Gamma$ ,  $\triangleleft: \Gamma \times F \rightarrow F$ , defined by  $sx = (x \triangleleft s)(x \triangleright s)$ ,  $x \in F$ ,  $s \in \Gamma$ .

Let  $\sigma \in Z^2(F, (k^{\Gamma})^{\times})$  and  $\tau \in Z^2(\Gamma, (k^F)^{\times})$  be normalized 2-cocycles with the respect to the actions afforded, respectively, by  $\triangleleft$  and  $\triangleright$ , subject to appropriate compatibility conditions [15].

The bicrossed product  $H = k^{\Gamma} \tau \#_{\sigma} kF$  associated to this data is a semisimple Hopf algebra. There is an *abelian* exact sequence

$$(2.5) \quad k \rightarrow k^{\Gamma} \rightarrow H \rightarrow kF \rightarrow k.$$

Moreover, every Hopf algebra fitting into such exact sequence can be described in this way. This gives a bijective correspondence between the equivalence classes of Hopf algebra extensions (2.5) associated to the matched pair  $(F, \Gamma)$  and a certain abelian group  $\text{Opext}(k^\Gamma, kF)$ .

*Remark 2.4.* The Hopf algebra  $H$  is group theoretical. In fact, we have an equivalence of fusion categories  $\text{Rep } H \simeq \mathcal{C}(G, \omega, F, 1)$  [19, 4.2], where  $\omega$  is the 3-cocycle on  $G$  coming from the so called *Kac exact sequence*.

Irreducible representations of  $H$  are classified by pairs  $(s, U_s)$ , where  $s$  runs over a set of representatives of the orbits of the action of  $F$  in  $\Gamma$ ,  $F_s = F \cap sFs^{-1}$  is the stabilizer of  $s \in \Gamma$ , and  $U_s$  is an irreducible representation of the twisted group algebra  $k_{\sigma_s} F_s$ , that is, an irreducible projective representation of  $F_s$  with cocycle  $\sigma_s$ , where  $\sigma_s(x, y) = \sigma(x, y)(s)$ ,  $x, y \in F$ ,  $s \in \Gamma$ . See [14].

Note that, for all  $s \in \Gamma$ , the restriction of  $\sigma_s : F \times F \rightarrow k^\times$  to the stabilizer  $F_s$  defines indeed a 2-cocycle on  $F_s$ .

The irreducible representation corresponding to such a pair  $(s, U_s)$  is in this case of the form

$$(2.6) \quad W_{(s, U_s)} := \text{Ind}_{k^\Gamma \otimes kF_s}^H s \otimes U_s.$$

**2.6. Quasitriangular Hopf algebras.** Let  $H$  be a finite dimensional Hopf algebra. Recall that  $H$  is called *quasitriangular* if there exists an invertible element  $R \in H \otimes H$ , called an *R-matrix*, such that

$$\begin{aligned} (\Delta \otimes \text{id})(R) &= R_{13}R_{23}, & (\epsilon \otimes \text{id})(R) &= 1, \\ (\text{id} \otimes \Delta)(R) &= R_{13}R_{12}, & (\text{id} \otimes \epsilon)(R) &= 1, \\ \Delta^{\text{cop}}(h) &= R\Delta(h)R^{-1}, & \forall h \in H. \end{aligned}$$

The existence of an *R-matrix* (also called a *quasitriangular structure* in what follows) amounts to the category  $\text{Rep } H$  being a braided tensor category. See [1].

For instance, the group algebra  $kG$  of a finite group  $G$  is a quasitriangular Hopf algebra with  $R = 1 \otimes 1$ . On the other hand, the dual Hopf algebra  $k^G$  admits a quasitriangular structure if and only if  $G$  is abelian.

If it exists, a quasitriangular structure in a Hopf algebra  $H$  need not be unique.

Another example of a quasitriangular Hopf algebra is the *Drinfeld double*  $D(H)$  of  $H$ , where  $H$  is any finite dimensional Hopf algebra. We have  $D(H) = H^{*\text{cop}} \otimes H$  as coalgebras, with a canonical *R-matrix*  $\mathcal{R} = \sum_i h^i \otimes h_i$ , where  $(h_i)_i$  is a basis of  $H$  and  $(h^i)_i$  is the dual basis.

As braided tensor categories,  $\text{Rep } D(H) = \mathcal{Z}(\text{Rep } H)$  is isomorphic to the center of the tensor category  $\text{Rep } H$ .



Suppose  $(H, R)$  is a quasitriangular Hopf algebra. There are Hopf algebra maps  $f_R : H^{*\text{cop}} \rightarrow H$  and  $f_{R_{21}} : H^* \rightarrow H^{\text{op}}$  defined by

$$f_R(p) = p(R^{(1)})R^{(2)}, \quad f_{R_{21}}(p) = p(R^{(2)})R^{(1)},$$

for all  $p \in H^*$ , where  $R = R^{(1)} \otimes R^{(2)} \in H \otimes H$ .

We shall denote  $f_R(H^*) = H_+$  and  $f_{R_{21}}(H^*) = H_-$ , respectively. So that  $H_+, H_-$  are Hopf subalgebras of  $H$  and we have  $H_+ \simeq (H_-^*)^{\text{cop}}$ .

We shall also denote by  $H_R = H_-H_+ = H_+H_-$  the minimal quasitriangular Hopf subalgebra of  $H$ . See [29].

By [29, Theorem 2], the multiplication of  $H$  determines a surjective Hopf algebra map  $D(H_-) \rightarrow H_R$ .

A quasitriangular Hopf algebra  $(H, R)$  is called *factorizable* if the map  $\Phi_R : H^* \rightarrow H$  is an isomorphism, where

$$(2.7) \quad \Phi_R(p) = p(Q^{(1)})Q^{(2)}, \quad p \in H^*;$$

here,  $Q = Q^{(1)} \otimes Q^{(2)} = R_{21}R \in H \otimes H$  [30].

If on the other hand  $\Phi_R = \epsilon 1$  (or equivalently,  $R_{21}R = 1 \otimes 1$ ), then  $(H, R)$  is called *triangular*. Finite dimensional triangular Hopf algebras were completely classified in [6]. In particular, if  $(H, R)$  is a semisimple quasitriangular Hopf algebra, then  $H$  is isomorphic, as a Hopf algebra, to a twisting  $(kG)^J$  of some finite group  $G$ .

It is well-known that the Drinfeld double  $(D(H), \mathcal{R})$  is indeed a *factorizable* quasitriangular Hopf algebra. We have  $D(H)_+ = H$ ,  $D(H)_- = H^{*\text{cop}}$ .

We shall use later on in this paper the following result about factorizable Hopf algebras. A categorical version is established in [11].

**Theorem 2.5.** [32, Theorem 2.3]. *Let  $(H, R)$  be a factorizable Hopf algebra. Then the map  $\Phi_R$  induces an isomorphism of groups  $G(H^*) \rightarrow G(H) \cap Z(H)$ .*

Note that we may identify  $G(D(H)) = G(H^*) \times G(H)$ . Under this identification, Theorem 2.5 gives us a group isomorphism  $G(D(H)^*) \rightarrow G(D(H)) \cap Z(D(H))$ , such that  $g \# f \mapsto f \# g$ . See also [29].

In particular, if  $f = \epsilon$ , then  $g \in G(H) \cap Z(H)$ , and also if  $g = 1$ , then  $f \in G(H^*) \cap Z(H^*)$ .

Suppose  $(H, R)$  is a finite dimensional quasitriangular Hopf algebra, and let  $D(H)$  be the Drinfeld double of  $H$ . Then there is a surjective Hopf algebra map  $f : D(H) \rightarrow H$ , such that  $(f \otimes f)\mathcal{R} = R$ . The map  $f$  is determined by  $f(p \otimes h) = f_R(p)h$ , for all  $p \in H^*$ ,  $h \in H$ .

This corresponds to the canonical inclusion of the braided tensor category  $\text{Rep } H$  (with braiding determined by the action of the  $R$ -matrix) into its center.

In particular, in the case where  $H$  is quasitriangular, the group  $G(H^*)$  can be identified with a subgroup of  $G(D(H)^*)$ .

## 3. NILPOTENCY

Let  $G$  be a finite group. A  $G$ -grading of a fusion category  $\mathcal{C}$  is a decomposition of  $\mathcal{C}$  as a direct sum of full abelian subcategories  $\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$ , such that  $\mathcal{C}_g^* = \mathcal{C}_{g^{-1}}$  and the tensor product  $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$  maps  $\mathcal{C}_g \times \mathcal{C}_h$  to  $\mathcal{C}_{gh}$ . The neutral component  $\mathcal{C}_e$  is thus a fusion subcategory of  $\mathcal{C}$ .

The grading is called *faithful* if  $\mathcal{C}_g \neq 0$ , for all  $g \in G$ . In this case,  $\mathcal{C}$  is called a  $G$ -extension of  $\mathcal{C}_e$  [8].

The following proposition is a consequence of [11, Theorem 3.8].

**Proposition 3.1.** *Let  $\mathcal{C} = \text{Rep } H$ , where  $H$  is a semisimple Hopf algebra. Then a faithful  $G$ -grading on  $\mathcal{C}$  corresponds to a central exact sequence of Hopf algebras  $k \rightarrow k^G \rightarrow H \rightarrow \overline{H} \rightarrow k$ , such that  $\text{Rep } \overline{H} = \mathcal{C}_e$ .*

Let  $\mathcal{C}$  be a fusion category and let  $\mathcal{C}_{\text{ad}}$  be the adjoint subcategory of  $\mathcal{C}$ . That is,  $\mathcal{C}_{\text{ad}}$  is the fusion subcategory of  $\mathcal{C}$  generated by  $X \otimes X^*$ , where  $X$  runs through the simple objects of  $\mathcal{C}$ .

It is shown in [11] that there is a canonical faithful grading on  $\mathcal{C}$ :  $\mathcal{C} = \bigoplus_{g \in U(\mathcal{C})} \mathcal{C}_g$ , called the *universal grading*, such that  $\mathcal{C}_e = \mathcal{C}_{\text{ad}}$ . The group  $U(\mathcal{C})$  is called the *universal grading group* of  $\mathcal{C}$ .

In the case where  $\mathcal{C} = \text{Rep } H$ , for a semisimple Hopf algebra  $H$ ,  $K = k^{U(\mathcal{C})}$  is the maximal central Hopf subalgebra of  $H$  and  $\mathcal{C}_{\text{ad}} = \text{Rep } H/HK^+$ .

Recall from [11, 8] that a fusion category  $\mathcal{C}$  is called (cyclically) *nilpotent* if there is a sequence of fusion categories

$$\mathcal{C}_0 = \text{Vec}, \mathcal{C}_1, \dots, \mathcal{C}_n = \mathcal{C}$$

and a sequence  $G_1, \dots, G_n$  of finite (cyclic) groups such that  $\mathcal{C}_i$  is faithfully graded by  $G_i$  with trivial component  $\mathcal{C}_{i-1}$ .

The semisimple Hopf algebra  $H$  is called nilpotent if the fusion category  $\text{Rep } H$  is nilpotent [11, Definition 4.4].

For instance, if  $G$  is a finite group, then the dual group algebra  $k^G$  is always nilpotent. However, the group algebra  $kG$  is nilpotent if and only if the group  $G$  is nilpotent [11, Remark 4.7. (1)].

**3.1. Nilpotency of an abelian extension.** It is shown in [10, Corollary 4.3] that a group-theoretical fusion category  $\mathcal{C}(G, \omega, F, \alpha)$  is nilpotent if and only if the normal closure of  $F$  in  $G$  is nilpotent. On the other hand, this happens if and only if  $F$  is nilpotent and subnormal in  $G$ , if and only if  $F \subseteq \text{Fit}(G)$ , where  $\text{Fit}(G)$  is the Fitting subgroup of  $G$ , that is, the unique largest normal nilpotent subgroup of  $G$  [10, Subsection 2.3].

Combined with Remark 2.4, this implies:

**Proposition 3.2.** *Let  $k \rightarrow k^\Gamma \rightarrow H \rightarrow kF \rightarrow k$  be an abelian exact sequence and let  $G = F \bowtie \Gamma$  be the associated factorizable group. Then  $H$  is nilpotent if and only if  $F \subseteq \text{Fit}(G)$ .*

An abelian exact sequence (2.5) is called *central* if the image of  $k^\Gamma$  is a central Hopf subalgebra of  $H$ . It is called *cocentral*, if the dual exact sequence is central.

The following facts are well-known:

**Lemma 3.3.** *Consider an abelian exact sequence (2.5). Then we have:*

- (i) *The sequence is central if and only if the action  $\triangleleft: \Gamma \times F \rightarrow \Gamma$  is trivial. In this case, the group  $G = F \rtimes \Gamma$  is a semidirect product  $G \simeq F \rtimes \Gamma$  with respect to the action  $\triangleright: \Gamma \times F \rightarrow F$ .*
- (ii) *The sequence is cocentral if and only if the action  $\triangleright: \Gamma \times F \rightarrow F$  is trivial. In this case, the group  $G = F \rtimes \Gamma$  is a semidirect product  $G \simeq F \rtimes \Gamma$  with respect to the action  $\triangleleft: \Gamma \times F \rightarrow \Gamma$ .  $\square$*

*Remark 3.4.* Assume the exact sequence (2.5) is central. Then  $F$  is a normal subgroup of  $G$ . It follows from Proposition 3.2 that  $H$  is nilpotent if and only if  $F$  is nilpotent.

#### 4. ON THE NILPOTENCY OF A CLASS OF SEMISIMPLE HOPF ALGEBRAS

Let  $p$  be a prime number. We shall consider in this subsection a nontrivial semisimple Hopf algebra  $H$  fitting into an abelian exact sequence

$$(4.1) \quad k \rightarrow k^{\mathbb{Z}_p} \rightarrow H \rightarrow kF \rightarrow k.$$

The main result of this subsection is Proposition 4.3 below.

We first have the following lemma.

Suppose that  $\mathcal{C}$  is any group-theoretical fusion category of the form  $\mathcal{C} = \mathcal{C}(G, \omega, \mathbb{Z}_p, \alpha)$  (Note that we may assume that  $\alpha = 1$ .) In particular,  $p$  divides the order of  $G(\mathcal{C})$ . We also have  $\text{c.d.}(\mathcal{C}) \subseteq \{1, p\}$ , by Corollary 2.2.

**Lemma 4.1.** *Let  $\mathcal{C} = \mathcal{C}(G, \omega, \mathbb{Z}_p, \alpha)$ . Assume that  $|G(\mathcal{C})| = p$ . Then  $G$  is a Frobenius group with Frobenius complement  $\mathbb{Z}_p$ .*

*Proof.* The description of the irreducible representations of  $\mathcal{C}$  in Subsection 2.4, combined with the assumption that  $|G(\mathcal{C})| = p$ , implies that  $g\mathbb{Z}_p g^{-1} \cap \mathbb{Z}_p = \{e\}$ , for all  $g \in G \setminus \mathbb{Z}_p$ . (In particular, the action of  $\mathbb{Z}_p$  on  $\mathbb{Z}_p \setminus G$  has no fixed points  $s \neq e$ .)

This condition says that  $G$  is a Frobenius group with Frobenius complement  $\mathbb{Z}_p$ , as claimed.  $\square$

*Remark 4.2.* Let  $G$  be as in Lemma 4.1. By Frobenius' Theorem we have that the Frobenius kernel  $N$  is a normal subgroup of  $G$ , such that  $G$  is a semidirect product  $G = N \rtimes \mathbb{Z}_p$ . Moreover,  $N$  is a nilpotent group, by a theorem of Thompson. See [31, Theorem 10.5.6], [12, Theorem (7.2)]. In fact, the Frobenius kernel  $N$  is equal to  $\text{Fit}(G)$ , the Fitting subgroup of  $G$  [31, Exercise 10.5.8].

As a consequence we get the following:

**Proposition 4.3.** *Consider the abelian exact sequence (4.1) and assume that  $|G(H)| = p$ . Then we have:*

- (i) *The sequence is central, that is,  $G(H) \subseteq Z(H)$ .*
- (ii)  *$G = F \rtimes \mathbb{Z}_p$  is a Frobenius group with kernel  $F$ . In particular,  $F$  is nilpotent.*

*Proof.* We follow the lines of the proof of [13, Proposition X.7 (i)]. Consider the matched pair  $(F, \mathbb{Z}_p)$  associated to (4.1), as in Subsection (2.5). Let  $G = F \rtimes \mathbb{Z}_p$  be the corresponding factorizable group.

Then  $\text{Rep } H^* \simeq \mathcal{C}(G, \omega, \mathbb{Z}_p, 1)$  is group-theoretical and by assumption,  $G(\text{Rep } H^*)$  is of order  $p$ . By Lemma 4.1,  $G$  is a Frobenius group with Frobenius complement  $\mathbb{Z}_p$ . Therefore  $G$  is a semidirect product  $G = N \rtimes \mathbb{Z}_p$ , where  $N = \text{Fit}(G)$  is a nilpotent subgroup.

Since  $|G(H)| = p$ , then the action of  $\mathbb{Z}_p$  on  $F$  has no fixed points. It follows, after decomposing  $F$  as a disjoint union of  $\mathbb{Z}_p$ -orbits, that  $|F| = 1 \pmod{p}$ . In particular,  $|F|$  is not divisible by  $p$ . Then  $F$  must map trivially under the canonical projection  $G \rightarrow G/N$ , that is,  $F \subseteq N$ . Hence  $F = N$ , because they have the same order. This shows (ii). Since  $F$  is normal in  $G$ , we get (i) in view of Lemma 3.3.  $\square$

**Corollary 4.4.** *Let  $k \rightarrow k^{\mathbb{Z}_p} \rightarrow H \rightarrow kF \rightarrow k$  be an abelian exact sequence such that  $|G(H)| = p$ . Then  $H$  is nilpotent.*

*Proof.* It follows from Proposition 4.3, in view of Remark 3.4.  $\square$

*Remark 4.5.* In view of [13, Theorem IX.8 (iii)], if  $H$  is a Kac algebra with  $\text{c.d.}(H^*) = \{1, p\}$  and  $|G(H)| = p$ , then  $H$  is a central abelian extension associated to an action of the cyclic group of order  $p$  on a nilpotent group. It follows from Corollary 4.4 that  $H$  is a nilpotent Hopf algebra.

*Remark 4.6.* Note that the (dual) assumption that  $\text{c.d.}(H) = \{1, p\}$  does not imply that  $H$  is nilpotent in general. For example, take  $H$  to be the group algebra of a nonabelian semidirect product  $F \rtimes \mathbb{Z}_p$ , where  $F$  is an abelian group such that  $(|F|, p) = 1$ .

On the other hand, the assumption on  $|G(H)|$  in Corollary 4.4 and Proposition 4.3 is essential. Namely, for all prime number  $p$ , there exist semisimple Hopf algebras  $H$  with  $\text{c.d.}(H^*) = \{1, p\}$  and such that  $H$  is *not* nilpotent.

To see an example, consider a group  $F$  with an automorphism of order  $p$  and suppose  $F$  is not nilpotent (take for instance  $F = \mathbb{S}_n$ , a symmetric group, such that  $n > 6$  is sufficiently large). Consider the corresponding action of  $\mathbb{Z}_p$  on  $F$  by group automorphisms and let  $G = F \rtimes \mathbb{Z}_p$  be the semidirect product.

Then there is an associated (split) abelian exact sequence  $k \rightarrow k^{\mathbb{Z}_p} \rightarrow H \rightarrow kF \rightarrow k$ , such that  $H$  is not commutative and not cocommutative. Moreover, in view of Remark 2.2,  $\text{c.d.}(H^*) = \{1, p\}$ . But, by Remark 3.4,  $H$  is not nilpotent, because  $F$  is not nilpotent by assumption.

**4.1. Reduction to abelian extensions from character degrees.** In this subsection we consider the case where  $\text{c.d.}(H) = \{1, p\}$  for some prime  $p$  and  $|G(H^*)| = p$ . We treat the problem of deducing an abelian extension like (4.1) from this assumption.

It is known, for instance, that if  $p = 2$ , then the assumption implies that  $H$  is cocommutative [2, Proposition 6.8], [13, Corollary IX.9].

**Lemma 4.7.** *Suppose that  $\text{c.d.}(H^*) = \{1, p\}$  for some prime  $p$ . Then  $H/(kG(H))^+H$  is a cocommutative coalgebra.*

*Proof.* Let  $\chi$  be an irreducible character of degree  $p$ . We have that

$$\chi\chi^* = \sum_{g \in G[\chi]} g + \sum_{\deg \lambda = p} \lambda.$$

So  $p \mid |G[\chi]|$ . Therefore  $|G[\chi]|$  is either  $p = \deg \chi$  or  $p^2$ , because it divides  $(\deg \chi)^2$ .

Moreover, since  $\chi = g\chi$  for all  $g \in G[\chi]$ , we have  $G[\chi]C = C$ , where  $C$  is the simple subcoalgebra of  $H$  containing  $\chi$ . Then it follows from [23, Remark 3.2.7] that  $C/(kG[\chi])^+C$  is a cocommutative coalgebra (indeed,  $|G[\chi]|$  is either  $p = \deg \chi$  or  $p^2$ , but in the last case,  $C/(kG[\chi])^+C$  is one-dimensional, hence also cocommutative). Then  $H/(kG(H))^+H$  is a cocommutative coalgebra, by [23, Corollary 3.3.2].  $\square$

**4.2. Results for the type  $(1, p; p, n)$ .** Let  $p$  be a prime number. Along this subsection  $H$  will be a semisimple Hopf algebra such that  $\text{c.d.}(H) = \{1, p\}$  and  $|G(H^*)| = p$ . So that  $H$  is of type  $(1, p; p, n)$  as an algebra.

**Proposition 4.8.** *Suppose that  $p$  divides  $|G(H)|$ . Then  $G(H^*) \subseteq Z(H^*)$  and  $H^*$  is nilpotent.*

*Proof.* By assumption, there is a subgroup  $G$  of  $G(H)$  with  $|G| = p$  (i.e.  $G \simeq \mathbb{Z}_p$ ) and the Hopf algebra inclusion  $kG \rightarrow H$  induces the following sequence:

$$kG(H^*) \xrightarrow{i} H^* \xrightarrow{\pi} kG,$$

with  $\pi$  surjective. By [23, Lemma 4.1.9], setting  $A = kG(H^*)$  and  $B = kG$ , we have that  $\pi \circ i : kG(H^*) \rightarrow kG$  is an isomorphism and  $H^* \simeq R \# kG(H^*) \simeq R \# \mathbb{Z}_p$  is a biproduct, where  $R \doteq (H^*)^{\text{co} \pi}$  is a semisimple braided Hopf algebra over  $\mathbb{Z}_p$ . The coalgebra  $R$  is cocommutative, by Lemma (4.7), because  $R \simeq H^*/H^*kG(H^*)^+$  as coalgebras. Since  $p \nmid 1 + mp = \dim R$  then by [34, Proposition 7.2],  $R$  is trivial. Therefore, by [23, Proposition 4.6.1],  $H^*$  fits into an abelian *central* exact sequence

$$k \rightarrow k\mathbb{Z}_p \rightarrow H^* \rightarrow R \rightarrow k.$$

Now, since the extension is abelian, there is a group  $F$  such that  $R \simeq kF$ . It follows from Corollary 4.4 that  $H^*$  is nilpotent.  $\square$

**Proposition 4.9.** *Suppose  $H$  is quasitriangular. Then  $G(H^*) \subseteq Z(H^*)$  and  $H^*$  is nilpotent.*

*Proof.* Consider the Drinfeld double  $D(H)$ . Since  $H$  is quasitriangular,  $G(H^*) \simeq \mathbb{Z}_p$  is isomorphic to a subgroup of  $G(D(H)^*)$ . Then  $G(D(H)^*)$  has an element  $g\#f$  of order  $p$ . We have that  $G(D(H)^*) \simeq G(D(H)) \cap Z(D(H)) \subseteq G(D(H)) = G(H^*) \times G(H)$ ; see Subsection 2.6.

In particular, the element  $f\#g \in G(D(H)) \cap Z(D(H))$  is of order  $p$ . If  $g$  is of order  $p$ , then the proposition follows from Proposition 4.8. Thus we may assume that  $g = 1$ . Then  $f \in G(H^*) \cap Z(H^*)$  is of order  $p$ , implying that  $G(H^*) \subseteq Z(H^*)$ .

Therefore  $H^*$  fits into an abelian central exact sequence  $k \rightarrow k^{\mathbb{Z}_p} \rightarrow H^* \rightarrow kF \rightarrow k$ , where  $F$  is a finite group such that  $kF \simeq H^*/H^*(k^{\mathbb{Z}_p})^+$ , by Lemma 4.7. In view of the assumption on the algebra structure of  $H$ , Corollary 4.4 implies that  $H^*$  is nilpotent, as claimed.  $\square$

**4.3. Results for the type  $(1, p; p, 1)$ .** We next discuss the case where  $H$  is of type  $(1, p; p, 1)$  as an algebra (not necessarily quasitriangular). In particular,  $\dim H = p(p+1)$  is even.

Notice that under this assumption, the category  $\text{Rep } H$  is a *near-group category* with fusion rule given by the group  $G = G(H^*) \simeq \mathbb{Z}_p$  and the integer  $\kappa$  [33].

Let  $\chi$  be the irreducible character of degree  $p$ . It follows that  $\chi = \chi^*$  and  $\chi g = \chi = g\chi$ . Then

$$\chi^2 = \sum_{g \in G(H^*)} g + \kappa\chi.$$

Taking degrees in the equation above we obtain  $p^2 = p + \kappa p$ , which means that  $\kappa = p - 1$ .

We shall use the following proposition. A more general statement will be proved in Theorem 6.2.

**Proposition 4.10.** *Suppose  $H$  is of type  $(1, p; p, 1)$  as an algebra. Then one of the following holds:*

- (i)  $p = 2$  and  $H \simeq k\mathbb{S}_3$ , or
- (ii)  $p = 2^\alpha - 1^1$ , and  $\dim H = 2^\alpha p$ .

*In particular,  $H$  is solvable.*

*Proof.* By [33, Theorem 1.2], it follows that  $G(H^*) \simeq \mathbb{Z}_{q^\alpha - 1}$ , for some prime  $q$  and  $\alpha \geq 1$ . Therefore  $p = q^\alpha - 1$ . If  $q > 2$ , then  $p = 2$ , which implies  $H \simeq k\mathbb{S}_3$  is cocommutative. If  $q = 2$ , then  $p$  has the particular expression  $p = 2^\alpha - 1$ .

Hence  $\dim H$  equals 6 or  $p(p+1) = 2^\alpha p$ . By Burnside's theorem for fusion categories [8, Theorem 1.6],  $H$  is solvable.  $\square$

*Remark 4.11.* Note that, for all prime number  $p$  such that  $p = 2$  or  $p = 2^\alpha - 1$  as in Proposition 4.10, then there exists a finite group whose group algebra has the prescribed algebra type, as follows from the construction in [33, Subsection 4.1].

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<sup>1</sup>Such a prime number is called a *Mersenne prime*, in particular  $\alpha$  must be prime.

**Proposition 4.12.** *Let  $H$  be a semisimple Hopf algebra of type  $(1, p; p, 1)$  as an algebra. Then  $G(H^*) \subseteq Z(H^*)$  and  $H^*$  is nilpotent.*

*Proof.* We have just proved in Proposition 4.10 that under this hypothesis  $H$  is solvable. Since  $\text{Rep } D(H) \simeq Z(\text{Rep } H)$ , then  $D(H)$  is also solvable [8, Proposition 4.5 (i)].

By [8, Proposition 4.5 (iv)],  $D(H)$  has nontrivial representations of dimension 1, that is,  $|G(D(H)^*)| \neq 1$ . We have that  $G(D(H)^*) \simeq G(D(H)) \cap Z(D(H)) \subseteq G(D(H)) = G(H^*) \times G(H)$ ; see Subsection 2.6.

We next argue as in the proof of Proposition 4.9. Consider an element  $1 \neq f \# g \in G(D(H)) \cap Z(D(H))$ . If  $f = 1$ , then  $1 \neq g \in Z(H) \cap G(H)$ . Therefore,  $H^*$  fits into a cocentral extension  $k \rightarrow K \rightarrow H^* \rightarrow k^{\langle g \rangle} \rightarrow k$ , where  $K$  is a *proper* normal Hopf subalgebra. The assumption on the algebra structure of  $H$  implies that  $K = kG(H^*)$ . Thus  $kG(H^*)$  is normal in  $H^*$ , and the extension is abelian, by Lemma 4.7. The proposition follows in this case from Proposition 4.3 (i) and Corollary 4.4.

Thus we may assume that  $f \neq 1$ . In particular,  $f$  is order  $p$ .

If  $|f| = |g| = p = |G(H^*)|$ , we have that  $p \mid |G(H)|$ . Then  $G(H^*) \subseteq Z(H^*)$  and  $H^*$  is nilpotent, by Proposition 4.8.

Otherwise, take  $|g| = n$ , with  $p \neq n$ . If  $f^n = 1$ , then  $p$  divides  $n$  and thus  $p$  divides  $|G(H)|$ . As before, we are done by Proposition 4.8.

If  $f^n \neq 1$ , then  $f^n \# 1 = (f^n \# g^n) = (f \# g)^n \in Z(D(H))$ , which implies that  $f^n \neq 1$  is central in  $H^*$  and thus  $G(H^*) \subseteq Z(H^*)$ .

Therefore  $H^*$  fits into an abelian central exact sequence  $k \rightarrow k^{\mathbb{Z}_p} \rightarrow H^* \rightarrow kF \rightarrow k$ , where  $F$  is a finite group such that  $kF \simeq H^*/H^*(k^{\mathbb{Z}_p})^+$ , by Lemma 4.7. In view of the assumption on the algebra structure of  $H$ , Corollary 4.4 implies that  $H^*$  is nilpotent, as claimed.  $\square$

**Theorem 4.13.** *Let  $H$  be a semisimple Hopf algebra of type  $(1, p, p, 1)$  as an algebra. Then  $H$  is isomorphic to a twisting of a group algebra  $(kN)^J$ , for some finite group  $N$  such that  $kN$  is of type  $(1, p, p, 1)$  as an algebra.*

*Proof.* We may assume that  $H$  is not cocommutative. Hence, in particular,  $p$  is odd. By Propositions 4.12 and 4.10,  $H^*$  fits into an abelian central exact sequence  $k \rightarrow k^{\mathbb{Z}_p} \rightarrow H^* \rightarrow kF \rightarrow k$ , where  $F$  is a finite group of order  $p + 1 = 2^\alpha$ . Then the action  $\triangleleft: \mathbb{Z}_p \times F \rightarrow \mathbb{Z}_p$  is trivial, while the action  $\triangleright: \mathbb{Z}_p \times F \rightarrow F$  is determined by an automorphism  $\varphi \in \text{Aut } F$  of order  $p = 2^\alpha - 1$ .

We first claim that the group  $F$  must be abelian. By a result of P. Hall [31, 5.3.3], since  $F$  is a 2-group, the order of  $\text{Aut } F$  divides the number  $n2^{(\alpha-r)r}$ , where  $n = |\text{GL}(r, 2)|$  and  $2^r$  equals the index in  $F$  of the Frattini subgroup  $\text{Frat}(F)$  (which is defined as the intersection of all the maximal subgroups of  $F$  [31, pp. 135]). In particular, we have  $r \leq \alpha$ .

Since the order of  $\varphi$  divides the order of  $\text{Aut } F$  and  $|\text{GL}(r, 2)| = (2^r - 1)(2^r - 2) \dots (2^r - 2^{r-1})$ , it follows that the prime  $p = 2^\alpha - 1$  divides  $2^r - 1$ , which means that  $r = \alpha$  and, therefore,  $\text{Frat}(F) = 1$ .

Since  $F$  is nilpotent (because it is a 2-group), a result of Wielandt [31, 5.2.16] implies that  $[F, F]$ , the commutator subgroup of  $F$ , is a subgroup of the Frattini subgroup  $\text{Frat}(F)$ . As we have just shown, we have  $\text{Frat}(F) = 1$  in this case. Thus  $[F, F] = 1$  and therefore  $F$  abelian, as claimed.

Consider the split extension  $B_0 = k^{\mathbb{Z}_p} \# kF$  associated to the matched pair  $(\mathbb{Z}_p, F)$ . Since  $F$  is abelian,  $B_0$  (being a central extension) is commutative. This means that  $B_0$  is isomorphic to  $k^N$ , where  $N = F \rtimes \mathbb{Z}_p$ .

Notice that  $|F| = 2^\alpha$  is relatively prime to  $p$ . It follows from [22, Proposition 5.22] and [16, Proposition 3.1] that  $H^*$  is obtained from the split extension  $B_0 = k^{\mathbb{Z}_p} \# kF \simeq k^N$  by twisting the multiplication. Indeed, the element representing the class of  $H^*$  in the group  $\text{Opext}(kF, k^{\mathbb{Z}_p})$  is the image of an element of  $H^2(F, k^\times)$  under the map  $H^2(F, k^\times) \oplus H^2(\mathbb{Z}_p, k^\times) \simeq H^2(F, k^\times) \rightarrow \text{Opext}(kF, k^{\mathbb{Z}_p})$  in the Kac exact sequence [16, Theorem 1.10]. Then the claim follows from [16, Proposition 3.1]. Dualizing, we get the statement in the theorem.  $\square$

*Remark 4.14.* If  $N$  is a group such that the group algebra  $kN$  is of type  $(1, p, p, 1)$  as an algebra and  $J \in kN \otimes kN$  is a twist, then  $H = (kN)^J$  is also of this type.

**Corollary 4.15.** *Let  $H$  be a semisimple Hopf algebra of type  $(1, p, p, 1)$  as an algebra. Then  $\text{Rep } H \simeq \text{Rep } N$  for some finite solvable group  $N$ .*

## 5. SOLVABILITY

Recall from [8] that a fusion category  $\mathcal{C}$  is called *weakly group-theoretical* if it is Morita equivalent to a nilpotent fusion category. If, furthermore,  $\mathcal{C}$  is Morita equivalent to a cyclically nilpotent fusion category, then  $\mathcal{C}$  is called *solvable*.

In other words,  $\mathcal{C}$  is weakly group-theoretical (solvable) if there exists an indecomposable algebra  $A$  in  $\mathcal{C}$  such that the category  ${}_A\mathcal{C}_A$  of  $A$ -bimodules in  $\mathcal{C}$  is a (cyclically) nilpotent fusion category.

Note that a group-theoretical fusion category is weakly group-theoretical.

On the other hand, the condition on  $\mathcal{C}$  being solvable is equivalent to the existence of a sequence of fusion categories

$$\mathcal{C}_0 = \text{Vec}_k, \mathcal{C}_1, \dots, \mathcal{C}_n = \mathcal{C},$$

such that  $\mathcal{C}_i$  is obtained from  $\mathcal{C}_{i-1}$  either by a  $G_i$ -equivariantization or as a  $G_i$ -extension, where  $G_1, \dots, G_n$  are cyclic groups of prime order. See [8, Proposition 4.4].

If  $G$  is a finite group and  $\omega \in H^3(G, k^\times)$ , we have that the categories  $\mathcal{C}(G, \omega)$  and  $\text{Rep } G$  are solvable if and only if  $G$  is solvable.

Let us call a semisimple Hopf algebra  $H$  *weakly group-theoretical* or *solvable*, if the category  $\text{Rep } H$  is weakly group-theoretical or solvable, respectively.



**5.1. Solvability of an abelian extension.** By [8, Proposition 4.5 (i)], solvability of a fusion category is preserved under Morita equivalence. Therefore, a group-theoretical fusion category  $\mathcal{C}(G, \omega, F, \alpha)$  is solvable if and only if the group  $G$  is solvable.

*Remark 5.1.* As a consequence of the Feit-Thompson theorem [9], we get that if the order of  $|G|$  is odd, then  $\mathcal{C}(G, \omega, F, \alpha)$  is solvable. This fact generalizes to weakly group-theoretical fusion categories; see Proposition 7.1 below.

This implies the following characterization of the solvability of an abelian extension:

**Corollary 5.2.** *Let  $H$  be a semisimple Hopf algebra fitting into an abelian exact sequence (2.5), then  $H$  is solvable if and only if  $G = F \bowtie \Gamma$  is solvable.*

In particular, if  $H$  is solvable, then  $F$  and  $\Gamma$  are solvable.

A result of Wielandt [36] implies that if the groups  $\Gamma$  and  $F$  are nilpotent, then  $G$  is solvable. As a consequence, we get the following:

**Corollary 5.3.** *Suppose  $\Gamma$  and  $F$  are nilpotent. Then  $H$  is solvable.*

Then, for instance, the abelian extensions in Proposition 4.3 are solvable.

Combining Corollary 5.3 with Lemma 4.1 and Remark 4.2, we get:

**Corollary 5.4.** *Let  $\mathcal{C} = \mathcal{C}(G, \omega, \mathbb{Z}_p, \alpha)$ . Assume that  $|G(\mathcal{C})| = p$ . Then  $\mathcal{C}$  is solvable.*

## 6. SOLVABILITY FROM CHARACTER DEGREES

Let  $p$  be a prime number. We study in this section fusion categories  $\mathcal{C}$  such that  $\text{c.d.}(\mathcal{C}) = \{1, p\}$ .

It is known that if  $G$  is a finite group, then this assumption implies that the group  $G$ , and thus the category  $\text{Rep } G$ , are solvable [12].

*Remark 6.1.* If  $H$  is any semisimple Hopf algebra such that  $\text{c.d.}(H) = \{1, p\}$  and  $G$  is any finite group, then the tensor product Hopf algebra  $A = H \otimes k^G$  also satisfies that  $\text{c.d.}(A) = \{1, p\}$  (since the irreducible modules of  $A$  are tensor products of irreducible modules of  $H$  and  $k^G$ ).

But  $A$  is not solvable unless  $G$  is solvable; indeed,  $k^G$  is a Hopf subalgebra as well as a quotient Hopf algebra of  $A$ .

Our aim in this section is to prove some structural results on  $\mathcal{C}$ , regarding solvability, under additional restrictions.

The following theorem generalizes Proposition 4.10.

**Theorem 6.2.** *Let  $\mathcal{C}$  be a near-group fusion category such that  $\text{c.d.}(\mathcal{C}) = \{1, p\}$ . Then  $\mathcal{C}$  is solvable.*

*Proof.* In the notation of [33], let the fusion rules of  $\mathcal{C}$  be given by the pair  $(G, \kappa)$ , where  $G$  is the group of invertible objects of  $\mathcal{C}$  and  $\kappa$  is a nonnegative integer. Then  $\text{Irr}(\mathcal{C}) = G \cup \{m\}$ , with the relation

$$(6.1) \quad m^2 = \sum_{g \in G} g + \kappa m.$$

The assumption on  $\text{c.d.}(\mathcal{C})$  implies that  $\text{FPdim } m = p$ . Hence  $\text{FPdim } \mathcal{C} = |G| + p^2$ , and since  $|G| = |G(\mathcal{C})|$  divides  $\text{FPdim } \mathcal{C}$ , we get that  $|G| = p$  or  $p^2$ . (Note that taking Frobenius-Perron dimensions in (6.1), we get that  $G \neq 1$ .)

If  $|G| = p^2$ , then  $\kappa = 0$  and  $\mathcal{C}$  is a Tambara-Yamagami category [35]. Furthermore,  $\mathcal{C}$  is a  $\mathbb{Z}_2$ -extension of a pointed category  $\mathcal{C}(G, \omega)$ . Then  $\mathcal{C}$  is solvable in this case, by [8, Proposition 4.5 (i)].

Suppose that  $|G| = p$ . Then  $\kappa = p - 1$ . As in the proof of Proposition 4.10, using [33, Theorem 1.2], we get that  $\text{FPdim } \mathcal{C} = p(p + 1)$  equals 6 or  $p2^\alpha$ . Then  $\mathcal{C}$  is solvable, by [8, Theorem 1.6].  $\square$

Our next result is the following theorem, for  $\mathcal{C} = \text{Rep } H$ , which is a consequence of Proposition 4.9. A stronger version of this result will be given in Subsection 7.2, under additional dimension restrictions.

**Theorem 6.3.** *Suppose  $H$  is of type  $(1, p; p, n)$  as an algebra. Assume in addition that  $H$  is quasitriangular. Then  $H$  is solvable.*

*Proof.* We have shown in Proposition 4.9 that  $H^*$  is nilpotent. Moreover, by Lemma 4.7,  $H$  fits into an abelian cocentral exact sequence  $k \rightarrow k^F \rightarrow H \rightarrow k\mathbb{Z}_p \rightarrow k$ , where  $F$  is a nilpotent group. Therefore,  $H$  is solvable, by Corollary 5.3.  $\square$

In the remaining of this section, we restrict ourselves to the case where  $\mathcal{C} = \text{Rep } H$  for a semisimple Hopf algebra  $H$ .

**6.1. The case  $p = 2$ .** Let  $H$  be a semisimple Hopf algebra such that  $\text{c.d.}(H) \subseteq \{1, 2\}$ . By [2, Theorem 6.4], one of the following possibilities holds:

- (i) There is a cocentral abelian exact sequence  $k \rightarrow k^F \rightarrow H \rightarrow k\Gamma \rightarrow k$ , where  $F$  is a finite group and  $\Gamma \simeq \mathbb{Z}_2^n$ ,  $n \geq 1$ , or
- (ii) There is a central exact sequence  $k \rightarrow k^U \rightarrow H \rightarrow B \rightarrow k$ , where  $B = H_{\text{ad}}$  is a proper Hopf algebra quotient, and  $U = U(\text{Rep } H)$  is the universal grading group of the category of finite dimensional  $H$ -modules.

In particular, if  $H = H_{\text{ad}}$ , then  $H$  satisfies (i).

As a consequence of this result we have:

**Theorem 6.4.** *Let  $H$  be a semisimple Hopf algebra such that  $\text{c.d.}(H) \subseteq \{1, 2\}$ . Then  $H$  is weakly group-theoretical.*

*Moreover, if  $H = H_{\text{ad}}$ , then  $H$  is group-theoretical.*

*Proof.* The assumption implies that  $H$  satisfies (i) or (ii) above. If  $H$  satisfies (i), then  $H$  is group-theoretical, by Remark 2.4.

Otherwise,  $H$  satisfies (ii), and then the category  $\text{Rep } H$  is a  $U$ -extension of  $\text{Rep } B$ , in view of Proposition 3.1. By an inductive argument, we may assume that  $B$  is weakly group-theoretical (note that  $\text{c.d.}(B) \subseteq \{1, 2\}$ ). Therefore so is  $H$ , by [8, Proposition 4.1].  $\square$

We next discuss conditions that guarantee the solvability of  $H$ . The following result is proved in [2].

**Proposition 6.5.** [2, Proposition 6.8]. *Suppose  $H$  is of type  $(1, 2; 2, n)$  as an algebra. Then  $H$  is cocommutative.*

The proposition implies that such a Hopf algebra  $H$  is isomorphic to a group algebra  $kG$  for some finite group  $G$ . By the assumption on the algebra structure of  $H$ , the group  $G$ , and then also  $H$ , are solvable.

The next lemma gives a sufficient condition for  $H$  to be solvable.

**Lemma 6.6.** *Suppose  $\text{c.d.}(H) \subseteq \{1, 2\}$  and  $H = H_{\text{ad}}$ . Then  $H$  is solvable if and only if the group  $F$  in (i) is solvable.*

*Proof.* Since  $H = H_{\text{ad}}$ , then  $H$  satisfies (i). Therefore  $H$  is solvable if and only if the relevant factorizable group  $G = F \bowtie \Gamma$  is solvable, by Corollary 5.2. Also, since the sequence (i) is cocentral, then  $G$  is a semidirect product:  $G = F \rtimes \Gamma$ . This proves the lemma.  $\square$

*Remark 6.7.* Suppose that  $H$  has a faithful irreducible character  $\chi$  of degree 2, such that  $\chi\chi^* = \chi^*\chi$ . Then it follows from [2, Theorem 3.5] that  $H$  fits into a central abelian exact sequence  $k \rightarrow k^{\mathbb{Z}^m} \rightarrow H \rightarrow kT \rightarrow k$ , for some polyhedral group  $T$  of even order and for some  $m \geq 1$ . In particular, since  $\text{c.d.}(H) = \{1, 2\}$ , then  $T$  is necessarily cyclic or dihedral (see, for instance, [2, pp. 10] for a description of the polyhedral groups and their character degrees). Therefore  $H$  is solvable in this case.

The assumption on  $\chi$  is satisfied in the case where  $H$  is quasitriangular; so that the conclusion holds in this case. We shall show in the next subsection that every quasitriangular semisimple Hopf algebra with  $\text{c.d.}(H) \subseteq \{1, 2\}$  is also solvable.

We next prove some lemmas that will be useful in the next subsection.

**Lemma 6.8.** *Suppose  $\text{c.d.}(H) \subseteq \{1, 2\}$  and let  $K$  be a Hopf subalgebra or quotient Hopf algebra of  $H$ . Then  $\text{c.d.}(K) \subseteq \{1, 2\}$ .*

*Proof.* We only need to show the claim when  $K \subseteq H$  is a Hopf subalgebra. In this case, the statement follows from surjectivity of the restriction functor  $\text{Rep } H \rightarrow \text{Rep } K$ .  $\square$

The lemma has the following immediate consequence:

**Corollary 6.9.** *If  $\text{c.d.}(H) \subseteq \{1, 2\}$ , then the group  $G(H)$  is solvable.*

**Lemma 6.10.** *Suppose  $\text{c.d.}(H), \text{c.d.}(H^*) \subseteq \{1, 2\}$ . Then  $H$  is solvable.*

*Proof.* By induction on the dimension of  $H$ .

Consider the universal grading group  $U$  of the category  $\text{Rep } H$ . Then  $H^* \rightarrow kU$  is a quotient Hopf algebra and therefore  $\text{c.d.}(U) \subseteq \{1, 2\}$ , by Lemma 6.8. This implies that the group  $U$  is solvable.

Suppose first  $H_{\text{ad}} \neq H$ . In view of Lemma 6.8, we also have  $\text{c.d.}(H_{\text{ad}}), \text{c.d.}(H_{\text{ad}}^*) \subseteq \{1, 2\}$ . By the inductive assumption  $H_{\text{ad}}$  is solvable. By [8, Proposition 4.5 (i)],  $H$  is solvable, since  $\text{Rep } H$  is a  $U$ -extension of  $\text{Rep } H_{\text{ad}}$ .

It remains to consider the case where  $H_{\text{ad}} = H$ . As pointed out at the beginning of this subsection, it follows from [2, Theorem 6.4] that in this case  $H$  satisfies condition (i), that is,  $H$  fits into a cocentral abelian exact sequence  $k \rightarrow k^F \rightarrow H \rightarrow k\Gamma \rightarrow k$ , with  $|\Gamma| > 1$  and  $\Gamma$  abelian.

In particular,  $k^\Gamma \subseteq H^*$  is a nontrivial central Hopf subalgebra, implying that  $H^* \neq H_{\text{ad}}^*$ . The inductive assumption implies, as before, that  $H_{\text{ad}}^*$  and thus also  $H^*$  is solvable. Then  $H$  also is. This finishes the proof of the lemma.  $\square$

**6.2. The quasitriangular case.** We shall assume in this subsection that  $H$  is quasitriangular. Let  $R \in H \otimes H$  be an  $R$ -matrix. We keep the notation in Subsection 2.6.

*Remark 6.11.* Notice that, since the category  $\text{Rep } H$  is braided, then the universal grading group  $U = U(\text{Rep } H)$  is abelian (and in particular, solvable).

The following is the main result of this subsection.

**Theorem 6.12.** *Let  $H$  be a quasitriangular semisimple Hopf algebra such that  $\text{c.d.}(H) \subseteq \{1, 2\}$ . Then  $H$  is solvable.*

*Proof.* If  $\text{c.d.}(H) = \{1\}$ , then  $H$  is commutative and, because it is quasitriangular, isomorphic to the group algebra of an abelian group. Hence we may assume that  $\text{c.d.}(H) = \{1, 2\}$ .

Consider the Hopf subalgebras  $H_+, H_- \subseteq H$ . By Lemma 6.8, we have  $\text{c.d.}(H_+), \text{c.d.}(H_-) \subseteq \{1, 2\}$ . Then  $\text{c.d.}(H_-), \text{c.d.}(H_-^*) \subseteq \{1, 2\}$ , since  $(H_-^*)^{\text{cop}} \simeq H_+$ .

By Lemma 6.10,  $H_-$  is solvable. Therefore the Drinfeld double  $D(H_-)$  and its homomorphic image  $H_R$  are also solvable.

We may thus assume that  $H_R \subsetneq H$ .

Observe that, being a quotient of  $H$ ,  $H_{\text{ad}}$  is also quasitriangular and satisfies  $\text{c.d.}(H_{\text{ad}}) \subseteq \{1, 2\}$ . Hence, by induction, we may also assume that  $H = H_{\text{ad}}$ , and in particular,  $G(H) \cap Z(H) = 1$ . Indeed,  $\text{Rep } H$  is a  $U$ -extension of  $\text{Rep } H_{\text{ad}}$  and the group  $U$  is abelian, as pointed out before.

Therefore  $H$  fits into a cocentral abelian exact sequence  $k \rightarrow k^F \rightarrow H \rightarrow k\Gamma \rightarrow k$ , where  $1 \neq \Gamma$  is elementary abelian of exponent 2.

In view of Lemma 6.6, it will be enough to show that the group  $F$  is solvable.

We have  $\widehat{\Gamma} \subseteq G(H^*) \cap Z(H^*)$ . By [28, Proposition 3],  $f_{R_{21}}(G(H^*) \cap Z(H^*)) \subseteq G(H) \cap Z(H)$ . Hence we may assume that  $f_{R_{21}}|_{\widehat{\Gamma}} = 1$  and similarly  $f_R|_{\widehat{\Gamma}} = 1$ . Thus  $f_R, f_{R_{21}}$  factorize through the quotient  $H^*/H^*(\widehat{\Gamma})^+ \simeq kF$ .

Therefore  $H_+ = f_R(H^*)$  and  $H_- = f_{R_{21}}(H^*)$  are cocommutative. (Then they are also commutative, since  $H_+ \simeq H_-^{\text{cop}}$ .) In particular,  $H_R = H_+H_-$  is cocommutative. Hence  $\Phi_R(H^*) \subseteq H_R \subseteq kG(H)$ .

By [21, Theorem 4.11],  $K = \Phi_R(H^*)$  is a commutative (and cocommutative) normal Hopf subalgebra, which is necessarily solvable, since  $H_R$  is. In addition,  $\Phi_R(H^*) \simeq kT$ , where  $T \subseteq G(H)$  is an abelian subgroup ([21, Example 2.1]), and there is an exact sequence of Hopf algebras

$$k \rightarrow kT \rightarrow H \xrightarrow{\pi} \overline{H} \rightarrow k,$$

where  $\overline{H}$  is a certain (canonical) triangular Hopf algebra.

Since  $\overline{H}$  is triangular, then  $\overline{H} \simeq (kL)^J$ , is a twisting of the group algebra of some finite group  $L$ . Because  $\text{c.d.}(L) = \text{c.d.}(\overline{H}) \subseteq \{1, 2\}$ ,  $L$  must be solvable. Hence  $\overline{H}$  is solvable, since  $\text{Rep } \overline{H} \simeq \text{Rep } L$ .

The map  $\pi : H \rightarrow \overline{H}$  induces, by restriction to the Hopf subalgebra  $k^F \subseteq H$ , an exact sequence

$$k \rightarrow kT \cap k^F \rightarrow k^F \xrightarrow{\pi|_{k^F}} \pi(k^F) \rightarrow k.$$

We have  $kT \cap k^F = k^{\overline{F}}$  and  $\pi(k^F) = k^S$ , where  $\overline{F}$  and  $S$  are a quotient and a subgroup of  $F$ , respectively, in such a way that the exact sequence above corresponds to an exact sequence of groups

$$1 \rightarrow S \rightarrow F \rightarrow \overline{F} \rightarrow 1.$$

Now,  $\overline{F}$  is abelian, because  $k^{\overline{F}} = kT \cap k^F$  is cocommutative, and  $S$  is solvable, because  $k^S$  is a Hopf subalgebra of  $\overline{H}$ . Therefore  $F$  is solvable. This implies that  $H$  is solvable and finishes the proof of the theorem.  $\square$

## 7. ODD DIMENSIONAL FUSION CATEGORIES

Along this section,  $p$  will be a prime number. Let  $\mathcal{C}$  be a fusion category over  $k$ . Recall that the set of irreducible degrees of  $\mathcal{C}$  was defined as

$$\text{c.d.}(\mathcal{C}) = \{\text{FPdim } x \mid x \in \text{Irr } \mathcal{C}\}.$$

The fusion categories that we shall consider in this section are all *integral*, that is, the Frobenius-Perron dimensions of objects of  $\mathcal{C}$  are (natural) integers. By [7, Theorem 8.33],  $\mathcal{C}$  is isomorphic to the category of representations of some finite dimensional semisimple *quasi*-Hopf algebra.

### 7.1. Odd dimensional weakly group-theoretical fusion categories.

The following result is a consequence of the Feit-Thompson theorem [9].

**Proposition 7.1.** *Let  $\mathcal{C}$  be a weakly group-theoretical fusion category and assume that  $\text{FPdim } \mathcal{C}$  is an odd integer. Then  $\mathcal{C}$  is solvable.*

Note that since  $\text{FPdim } \mathcal{C}$  is an odd integer, the fusion category  $\mathcal{C}$  is integral. See [5, Corollary 2.22].

*Proof.* By definition,  $\mathcal{C}$  is Morita equivalent to a nilpotent fusion category. Then, by [8, Proposition 4.5 (i)], it will be enough to show that a *nilpotent* fusion category of odd Frobenius-Perron dimension is solvable. So, assume that  $\mathcal{C}$  is nilpotent, so that  $\mathcal{C}$  is a  $G$ -extension of a fusion subcategory  $\tilde{\mathcal{C}}$ , with  $|G| > 1$ . In particular,  $\text{FPdim } \mathcal{C} = |G| \text{FPdim } \tilde{\mathcal{C}}$ . Hence the order of  $G$  and  $\text{FPdim } \tilde{\mathcal{C}}$  are odd and  $\text{FPdim } \tilde{\mathcal{C}} < \text{FPdim } \mathcal{C}$ . The theorem follows by induction, since by the Feit-Thompson theorem,  $G$  is solvable. See [8, Proposition 4.5 (i)].  $\square$

**7.2. Braided fusion categories.** We shall need the following lemma whose proof is contained in the proof of [8, Proposition 6.2 (i)]. We include a sketch of the argument for the sake of completeness.

**Lemma 7.2.** *Let  $\mathcal{C}$  be a fusion category and let  $G$  be a finite group acting on  $\mathcal{C}$  by tensor autoequivalences. Assume  $\text{c.d.}(\mathcal{C}^G) \subseteq \{1, p\}$ . Then  $\text{c.d.}(\mathcal{C}) \subseteq \{1, p\}$ .*

*Proof.* Regard  $\mathcal{C}$  as an indecomposable module category over itself via tensor product, and similarly for  $\mathcal{C}^G$ . Let  $Y$  be a simple object of  $\mathcal{C}$ . Since the forgetful functor  $F : \mathcal{C}^G \rightarrow \mathcal{C}$  is surjective,  $Y$  is a simple constituent of  $F(X)$ , for some simple object  $X$  of  $\mathcal{C}^G$ .

Since  $F$  is a tensor functor, we have  $\text{FPdim } X = \text{FPdim } F(X)$ . By Formula (7) in [8, Proof of Proposition 6.2],

$$(7.1) \quad \text{FPdim}(X) = \deg(\pi)[G : G_Y] \text{FPdim } Y,$$

where  $G_Y \subseteq G$  is the stabilizer of  $Y$  and  $\pi$  is an irreducible representation of  $G_Y$  associated to  $X$ . Therefore  $\text{FPdim } Y$  divides  $\text{FPdim } X$ .

The assumption on  $\mathcal{C}^G$  implies that  $\text{FPdim } X = 1$  or  $p$ . Then also  $\text{FPdim } Y = 1$  or  $p$ . This proves the lemma.  $\square$

**Theorem 7.3.** *Let  $\mathcal{C}$  be braided fusion category such that  $\text{FPdim } \mathcal{C}$  is an odd natural integer. Assume in addition that  $\text{c.d.}(\mathcal{C}) \subseteq \{1, p\}$ . Then  $\mathcal{C}$  is solvable.*

*Proof.* By induction on  $\text{FPdim } \mathcal{C}$ . (Notice that the Frobenius-Perron dimension of a fusion subcategory of  $\mathcal{C}$  divides the dimension of  $\mathcal{C}$  [7, Proposition 8.15], and the same is true for the Frobenius-Perron dimension of a fusion category  $\mathcal{D}$  such that there exists a surjective tensor functor  $\mathcal{C} \rightarrow \mathcal{D}$  [7, Corollary 8.11]. Thus these fusion categories are odd-dimensional as well.) If  $\text{c.d.}(\mathcal{C}) = \{1\}$ , then  $\mathcal{C}$  is pointed. Then  $\mathcal{C} \simeq \mathcal{C}(G, \omega)$  for some group  $G$  of order  $\text{FPdim } \mathcal{C}$  and some 3-cocycle  $\omega$  on  $G$ . Then the result follows from [8, Proposition 4.5 (ii)] and the Feit-Thompson theorem, since  $|G| = \text{FPdim } \mathcal{C}$  is odd.

Suppose next that  $\text{c.d.}(\mathcal{C}) = \{1, p\}$ . Then all non-invertible objects in  $\mathcal{C}$  have Frobenius-Perron dimension  $p$ . Consider the group  $G(\mathcal{C})$  of invertible objects of  $\mathcal{C}$ . Then  $G(\mathcal{C}) \neq 1$ , as follows by taking Frobenius-Perron dimensions in a decomposition of the tensor product  $X \otimes X^*$ , for some simple non-invertible object  $X$ . Moreover, the order of  $G(\mathcal{C})$  is odd (because it equals  $\text{FPdim } \mathcal{C}_{pt}$  and hence divides  $\text{FPdim } \mathcal{C}$ ) and therefore  $G(\mathcal{C})$  is solvable.

The assumption on  $\text{FPdim } \mathcal{C}$  implies that  $\mathcal{C}$  (regarded with its canonical spherical structure) is modularizable, in view of [4, Lemma 7.2]. Let  $\tilde{\mathcal{C}}$  be its modularization, which is a modular category over  $k$ . Then  $\mathcal{C}$  is an equivariantization  $\mathcal{C} \simeq \tilde{\mathcal{C}}^G$  with respect to the action of a certain group  $G$  on  $\tilde{\mathcal{C}}$  [3]. (Indeed, the modularization functor  $\mathcal{C} \rightarrow \tilde{\mathcal{C}}$  gives rise to an exact sequence of fusion categories  $\text{Rep } G \rightarrow \mathcal{C} \rightarrow \tilde{\mathcal{C}}$ , which comes from an equivariantization; see [4, Example 5.33].)

By construction of  $G$ , the category  $\text{Rep } G$  is the (tannakian) fusion subcategory of transparent objects in  $\mathcal{C}$ . Therefore there is an embedding of braided fusion categories  $\text{Rep } G \subseteq \mathcal{C}$ . In particular, the order of  $G$  is odd, implying that  $G$  is solvable.

By Lemma 7.2,  $\text{c.d.}(\tilde{\mathcal{C}}) \subseteq \{1, p\}$ . Then, by induction, and since an equivariantization of a solvable fusion category under the action of a solvable group is again solvable, we may and shall assume in what follows that  $\mathcal{C} = \tilde{\mathcal{C}}$  is modular.

It is shown in [11, Theorem 6.2] that the universal grading group  $U(\mathcal{C})$  is (abelian and) isomorphic to the group  $\widehat{G(\mathcal{C})}$  of characters of  $G(\mathcal{C})$ . In particular,  $U(\mathcal{C}) \neq 1$ . On the other hand,  $\mathcal{C}$  is a  $U(\mathcal{C})$ -extension of its fusion subcategory  $\mathcal{C}_{ad}$ . Since also  $\text{c.d.}(\mathcal{C}_{ad}) \subseteq \{1, p\}$ , then  $\mathcal{C}_{ad}$  is solvable, by induction. Therefore  $\mathcal{C}$  is solvable, as claimed.  $\square$

## REFERENCES

- [1] B. BAKALOV and A. KIRILLOV JR. *Lectures on Tensor categories and modular functors*, University Lecture Series **21**, Am. Math. Soc., Providence, 2001.
- [2] J. BICHON and S. NATALE, *Hopf algebra deformations of binary polyhedral groups*, to appear in Transf. Groups, preprint [arXiv:0907.1879v1](https://arxiv.org/abs/0907.1879).
- [3] A. BRUGUIÈRES, *Catégories prémodulaires, modularisations et invariants des variétés de dimension 3*, Math. Ann. **316**, 215–236 (2000).
- [4] A. BRUGUIÈRES and S. NATALE, *Exact sequences of tensor categories*, Int. Math. Res. Not. doi:10.1093/imrn/rnq294, 1–62 (2011).
- [5] V. DRINFELD, S. GELAKI, D. NIKSHYCH and V. OSTRIK, *On braided fusion categories I*, Sel. Math. New Ser. **16**, 1–119 (2010).
- [6] P. ETINGOF and S. GELAKI, *The classification of finite dimensional triangular Hopf algebras over an algebraically closed field of characteristic 0*, Mosc. Math. J. **3**, 37–43 (2003).
- [7] P. ETINGOF, D. NIKSHYCH and V. OSTRIK, *On fusion categories*, Annals of Mathematics **162**, 581–642 (2005).
- [8] P. ETINGOF, D. NIKSHYCH and V. OSTRIK, *Weakly group-theoretical and solvable fusion categories*, Adv. Math. **226**, 176–205 (2011).

- [9] W. FEIT and J. THOMPSON, *Solvability of groups of odd order*, Pacific J. Math. **13**, 775–1029 (1963).
- [10] S. GELAKI and D. NAIDU, *Some properties of group-theoretical categories*, J. Algebra **322**, 2631–2641 (2009).
- [11] S. GELAKI and D. NIKSHYCH, *Nilpotent fusion categories*, Adv. Math. **217**, 1053–1071 (2008).
- [12] I. ISAACS, *Character theory of finite groups*, Pure and Applied Mathematics **69**, Academic Press, New York, 1976.
- [13] M. IZUMI and H. KOSAKI *Kac algebras arising from composition of subfactors: general theory and classification*, Mem. Amer. Math. Soc. **158**, 750, (2007).
- [14] Y. KASHINA, G. MASON and S. MONTGOMERY, *Computing the Frobenius-Schur indicator for abelian extensions of Hopf algebras*, J. Algebra **251**, 888–913 (2002).
- [15] A. MASUOKA, *Extensions of Hopf algebras*, Trabajos de Matemática **41/99**, Universidad Nacional de Córdoba, (1999).
- [16] A. MASUOKA, *Hopf algebra extensions and cohomology*, Math. Sci. Res. Inst. Publ. **43**, 167–209 (2002).
- [17] S. MONTGOMERY and S. WHITERSPOON, *Irreducible representations of crossed products*, J. Pure Appl. Algebra **129**, 315–326 (1998).
- [18] S. NATALE, *On semisimple Hopf algebras of dimension  $pq^2$* , J. Algebra **221**, 242–278 (1999).
- [19] S. NATALE, *On group-theoretical Hopf algebras and exact factorizations of finite groups*, J. Algebra **270**, 199–211 (2003).
- [20] S. NATALE, *Frobenius-Schur indicators for a class of fusion categories*, Pacific J. Math. **221**, 353–378 (2005).
- [21] S. NATALE, *R-matrices and Hopf algebra quotients*, Int. Math. Res. Not. **2006**, 1–18 (2006).
- [22] S. NATALE, *On the exponent of tensor categories coming from finite groups*, Israel J. Math. **162**, 253–273 (2007).
- [23] S. NATALE, *Semisolvability of Semisimple Hopf Algebras of Low Dimension*, Mem. Amer. Math. Soc. **186**, 874, (2007).
- [24] S. NATALE, *Semisimple Hopf algebras and their representations*, to appear in Publ. Mat. Uruguay (2010).
- [25] W. NICHOLS and M. RICHMOND, *The Grothendieck group of a Hopf algebra*, J. Pure Appl. Algebra **106**, 297–306 (1996).
- [26] W. NICHOLS and M. ZOELLER, *A Hopf algebra freeness Theorem*, Amer. J. Math. **111**, 381–385 (1989).
- [27] V. OSTRIK, *Module categories over the Drinfeld double of a finite group*, Int. Math. Res. Not. **2003**, 1507–1520 (2003).
- [28] D. RADFORD, *On the antipode of a quasitriangular Hopf algebra*, J. Algebra **151**, 1–11 (1992).
- [29] D. RADFORD, *Minimal quasitriangular Hopf algebras*, J. Algebra **157**, 285–315 (1993).
- [30] N. RESHETIKHIN and M. SEMENOV-TIAN-SHANSKY *Quantum R-matrices and factorization problems*, J. Geom. Phys. **5**, 533–550 (1988).
- [31] D. ROBINSON, *A course in the theory of groups*, Graduate Texts Math. **80**, Springer-Verlag, Berlin, 1982.
- [32] H.-J. SCHNEIDER, *Some properties of factorizable Hopf algebras*, Proc. Amer. Math. Soc. **29**, 1891–1898 (2001).
- [33] J. SIEHLER, *Near-group categories*, Algebr. Geom. Topol. **3**, 719–775 (2003).
- [34] Y. SOMMERHÄUSER, *Yetter-Drinfel'd Hopf algebras over groups of prime order*, Lectures Notes in Math. **1789**, Springer-Verlag (2002).
- [35] D. TAMBARA and S. YAMAGAMI, *Tensor categories with fusion rules of self-duality for finite abelian groups*, J. Algebra **209**, 692–707 (1998).



- [36] H. WIELANDT, *Über Produkte von nilpotenten Gruppen*, Illinois J. Math. **2**, 611-618 (1958).
- [37] S. ZHU, *On finite dimensional semisimple Hopf algebras*, Commun. Algebra **21**, 3871-3885 (1993).

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# ON FUSION CATEGORIES WITH FEW IRREDUCIBLE DEGREES

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**ABSTRACT.** We prove some results on the structure of certain classes of integral fusion categories and semisimple Hopf algebras under restrictions on the set of its irreducible degrees.

## 1. INTRODUCTION

Let  $k$  be an algebraically closed field of characteristic zero. Let  $\mathcal{C}$  be a fusion category over  $k$ . That is,  $\mathcal{C}$  is a  $k$ -linear semisimple rigid tensor category with finitely many isomorphism classes of simple objects, finite-dimensional spaces of morphisms, and such that the unit object  $\mathbf{1}$  of  $\mathcal{C}$  is simple.

For example, if  $G$  is a finite group, then the categories  $\text{Rep } G$  of its finite dimensional representations, and the category  $\mathcal{C}(G, \omega)$  of  $G$ -graded vector spaces with associativity determined by the 3-cocycle  $\omega$ , are fusion categories over  $k$ . More generally, if  $H$  is a finite dimensional semisimple quasi-Hopf algebra over  $k$ , then the category  $\text{Rep } H$  of its finite dimensional representations is a fusion category.

Let  $\text{Irr}(\mathcal{C})$  denote the set of isomorphism classes of simple objects in the fusion category  $\mathcal{C}$ . In analogy with the case of finite groups [12], we shall use the notation  $\text{c.d.}(\mathcal{C})$  to indicate the set

$$\text{c.d.}(\mathcal{C}) = \{\text{FPdim } x \mid x \in \text{Irr}(\mathcal{C})\}.$$

Here,  $\text{FPdim } x$  denotes the *Frobenius-Perron dimension* of  $x \in \text{Irr}(\mathcal{C})$ . Notice that, when  $\mathcal{C}$  is the representation category of a quasi-Hopf algebra, Frobenius-Perron dimensions coincide with the dimensions of the underlying vector spaces. In this case, we shall use the notation  $\text{c.d.}(\mathcal{C}) = \text{c.d.}(H)$ .

The positive real numbers  $\text{FPdim } x$ ,  $x \in \text{Irr}(\mathcal{C})$ , will be called the *irreducible degrees* of  $\mathcal{C}$ .

The fusion categories that we shall consider in this paper are all *integral*, that is, the Frobenius-Perron dimensions of objects of  $\mathcal{C}$  are (natural) integers. By [7, Theorem 8.33],  $\mathcal{C}$  is isomorphic to the category of representations of some finite dimensional semisimple quasi-Hopf algebra.

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For a finite group  $G$ , the knowledge of the set  $\text{c.d.}(G) = \text{c.d.}(kG)$  gives in some cases substantial information about the structure of  $G$ . It is known, for instance, that if  $|\text{c.d.}(G)| \leq 3$ , then  $G$  is solvable.

On the other hand, if  $|\text{c.d.}(G)| = 2$ , say  $\text{c.d.}(G) = \{1, m\}$ ,  $m \geq 1$ , then either  $G$  has an abelian normal subgroup of index  $m$  or else  $G$  is nilpotent of class  $\leq 3$ . Furthermore, if  $G$  is nonabelian, then  $\text{c.d.}(G) = \{1, p\}$  for some prime number  $p$ , if and only if  $G$  contains an abelian normal subgroup of index  $p$  or the center  $Z(G)$  has index  $p^3$ . See [12, Theorems (12.11), (12.14) and (12.15)].

In the context of semisimple Hopf algebras, some results in the same spirit are known. A basic one is that of [39], which asserts that if  $|\text{c.d.}(H)| \leq 3$ , then  $G(H^*)$  is not trivial, in other words,  $H$  has nontrivial characters of degree 1. A similar result appears in [19, Theorem 2.2.3].

Further results, leading to classification theorems in some specific cases, appear in the work of Izumi and Kosaki [13] for Kac algebras, that is, Hopf  $C^*$ -algebras.

In this paper we consider the general problem of understanding the structure of a fusion category  $\mathcal{C}$  after the knowledge of  $\text{c.d.}(\mathcal{C})$ . For instance, it is well-known that  $\text{c.d.}(\mathcal{C}) = \{1\}$  if and only if  $\mathcal{C}$  is pointed, if and only if  $\mathcal{C} \simeq \mathcal{C}(G, \omega)$ , for some 3-cocycle  $\omega$  on the group  $G = G(\mathcal{C})$  of isomorphism classes of invertible objects of  $\mathcal{C}$ .

More specifically, we address the following question:

**Question 1.1.** *Suppose  $\text{c.d.}(\mathcal{C}) = \{1, p\}$ , with  $p$  a prime number. What can be said about the structure of  $\mathcal{C}$ ?*

We treat mostly structural questions regarding nilpotency and solvability, in the sense introduced in [11] and [8]. (A related question for semisimple Hopf algebras, that we shall not discuss in the present paper, was posed in [25, Question 7.2].)

The notions of nilpotency and solvability of a fusion category are related to the corresponding notions for finite groups as follows: if  $G$  is a finite group, then the category  $\text{Rep } G$  is nilpotent or solvable if and only if  $G$  is nilpotent or solvable, respectively. On the dual side, a pointed fusion category  $\mathcal{C}(G, \omega)$  is always nilpotent, while it is solvable if and only if the group  $G$  is solvable.

An important class of fusion categories, called *weakly group-theoretical* fusion categories, was introduced and studied in [8]. This generalized in turn the notion of a group-theoretical fusion category of [7]. By definition,  $\mathcal{C}$  is group-theoretical if it is Morita equivalent to a pointed fusion category, and it is weakly group-theoretical if it is Morita equivalent to a nilpotent fusion category. Every nilpotent or solvable fusion category is weakly group-theoretical.

With regard to Question 1.1, consider for instance the case where  $\mathcal{C} = \text{Rep } H$ , for a semisimple Hopf algebra  $H$ . A result in this direction is known

in the case  $p = 2$ . It is shown in [2, Corollary 6.6] that if  $H$  is a semisimple Hopf algebra such that  $\text{c.d.}(H) \subseteq \{1, 2\}$ , then  $H$  is upper semisolvable. Moreover,  $H$  is necessarily cocommutative if  $G(H^*)$  is of order 2. The proof of these results relies on a refinement of a theorem of Nichols and Richmond ([26, Theorem 11]) given in [2, Theorem 1.1].

In the context of Kac algebras, it is shown in [13, Theorem IX.8 (iii)] that if  $\text{c.d.}(H^*) = \{1, p\}$  and in addition  $|G(H)| = p$ , then  $H$  is a central abelian extension associated to an action of the cyclic group of order  $p$  on a nilpotent group. In the recent terminology introduced in [11], this result implies that such a Kac algebra is *nilpotent*. See Remark 4.5.

The main results of this paper are summarized in the following theorem.

**Theorem 1.2.** *Let  $\mathcal{C}$  be a fusion category over  $k$ . Then we have:*

(i) *Suppose  $\mathcal{C}$  is weakly group-theoretical and has odd dimension. Then  $\mathcal{C}$  is solvable. (Proposition 7.1.)*

*Let  $p$  be a prime number.*

(ii) *Suppose  $\mathcal{C}$  is odd-dimensional and  $\text{c.d.}(\mathcal{C}) \subseteq \{p^m : m \geq 0\}$ . Then  $\mathcal{C}$  is solvable. (Theorem 7.3.)*

(iii) *Suppose  $\text{c.d.}(\mathcal{C}) \subseteq \{1, p\}$ . Then  $\mathcal{C}$  is solvable in any of the following cases:*

- $\mathcal{C} = \mathcal{C}(G, \omega, \mathbb{Z}_p, \alpha)$  (a group-theoretical fusion category [7]) and  $G(\mathcal{C})$  is of order  $p$ . (Corollary 5.4.)
- $\mathcal{C}$  is a near-group category [35]. (Theorem 6.2.)
- $\mathcal{C} = \text{Rep } H$ , where  $H$  is a semisimple quasitriangular Hopf algebra and  $p = 2$ . (Theorem 6.12.)

(iv) *Let  $H$  be a semisimple Hopf algebra such that  $\text{c.d.}(H) \subseteq \{1, p\}$ . Then  $H^*$  is nilpotent in any of the following cases:*

- $|G(H^*)| = p$  and  $p$  divides  $|G(H)|$ . (Proposition 4.8.)
- $|G(H^*)| = p$  and  $H$  is quasitriangular. (Proposition 4.9.)
- $H$  is of type  $(1, p; p, 1)$  as an algebra. (Proposition 4.12.)

(v) *Let  $H$  be a semisimple Hopf algebra such that  $\text{c.d.}(H) \subseteq \{1, 2\}$ . Then we have:*

- $H$  is weakly group-theoretical, and furthermore, it is group-theoretical if  $H = H_{\text{ad}}$ . (Theorem 6.4.)
- The group  $G(H)$  is solvable. (Corollary 6.9.)

(vi) *Let  $H$  be a semisimple Hopf algebra of type  $(1, p; p, 1)$  as an algebra. Then  $H$  is isomorphic to a twisting of the group algebra  $kN$ , where either  $p = 2$  and  $N = \mathbb{S}_3$  or  $p = 2^{\alpha-1}$ ,  $\alpha > 1$ , and  $N$  is the affine group of the field  $\mathbb{F}_{2^\alpha}$ . (Theorem 4.13.)*

The proof of part (i) of the theorem is a consequence of the Feit-Thompson Theorem [9] that asserts that every finite group of odd order is solvable.

By [25, Corollary 4.5], the semisimple Hopf algebras  $H$  in part (iv) of the theorem are *lower semisolvable* in the sense of [18].

The results on semisimple Hopf algebras  $H$  with  $\text{c.d.}(H) \subseteq \{1, 2\}$  rely on the results of the paper [2]. We also make strong use of several results of the papers [8], [11] and [10] on weakly group-theoretical, solvable and nilpotent fusion categories.

The paper is organized as follows. In Section 2 we recall the main notions and results relevant to the problem we consider. In particular, several properties of group-theoretical fusion categories and Hopf algebra extensions are discussed here. The results on nilpotency are contained in Sections 3 and 4. The strategy in these sections consists in reducing the problem to considering Hopf algebra extensions. Sections 5, 6 and 7 are devoted to the solvability question in different situations.

## 2. PRELIMINARIES

**2.1. Fusion categories.** A *fusion category* over  $k$  is a  $k$ -linear semisimple rigid tensor category  $\mathcal{C}$  with finitely many isomorphism classes of simple objects, finite-dimensional spaces of morphisms, and such that the unit object  $\mathbf{1}$  of  $\mathcal{C}$  is simple. We refer the reader to [1, 7] for basic definitions and facts concerning fusion categories. In particular, if  $H$  is a semisimple (quasi-)Hopf algebra over  $k$ , then  $\text{Rep } H$  is a fusion category.

A *fusion subcategory* of a fusion category  $\mathcal{C}$  is a full tensor subcategory  $\mathcal{C}' \subseteq \mathcal{C}$  such that if  $X \in \mathcal{C}$  is isomorphic to a direct summand of an object of  $\mathcal{C}'$ , then  $X \in \mathcal{C}'$ . A fusion subcategory is necessarily rigid, so it is indeed a fusion category [5, Corollary F.7 (i)].

A *pointed fusion category* is a fusion category where all simple objects are invertible. A pointed fusion category is equivalent to the category  $\mathcal{C}(G, \omega)$ , of finite-dimensional  $G$ -graded vector spaces with associativity constraint determined by a cohomology class  $\omega \in H^3(G, k^\times)$ , for some finite group  $G$ . In other words,  $\mathcal{C}(G, \omega)$  is the category of representations of the quasi-Hopf algebra  $k^G$ , with associator  $\omega \in (k^G)^{\otimes 3}$ .

The fusion subcategory *generated* by a collection  $\mathcal{X}$  of objects of  $\mathcal{C}$  is the smallest fusion subcategory containing  $\mathcal{X}$ .

If  $\mathcal{C}$  is a fusion category, then the set of isomorphism classes of invertible objects of  $\mathcal{C}$  forms a group, denoted  $G(\mathcal{C})$ . The fusion subcategory generated by the invertible objects of  $\mathcal{C}$  is a fusion subcategory, denoted  $\mathcal{C}_{pt}$ ; it is the maximal pointed subcategory of  $\mathcal{C}$ .

Let  $\text{Irr}(\mathcal{C})$  denote the set of isomorphism classes of simple objects in the fusion category  $\mathcal{C}$ . The set  $\text{Irr}(\mathcal{C})$  is a basis over  $\mathbb{Z}$  of the Grothendieck ring  $\mathcal{G}(\mathcal{C})$ .

**2.2. Irreducible degrees.** For  $x \in \text{Irr}(\mathcal{C})$ , let  $\text{FPdim } x$  be its Frobenius-Perron dimension. The positive real numbers  $\text{FPdim } x$ ,  $x \in \text{Irr}(\mathcal{C})$  will be called the *irreducible degrees* of  $\mathcal{C}$ . These extend to a ring homomorphism  $\text{FPdim} : \mathcal{G}(\mathcal{C}) \rightarrow \mathbb{R}$ . When  $\mathcal{C}$  is the representation category of a quasi-Hopf algebra, Frobenius-Perron dimensions coincide with the dimension of the underlying vector spaces.

The set of *irreducible degrees* of  $\mathcal{C}$  is defined as

$$\text{c.d.}(\mathcal{C}) = \{\text{FPdim } x \mid x \in \text{Irr}(\mathcal{C})\}.$$

The category  $\mathcal{C}$  is called *integral* if  $\text{c.d.}(\mathcal{C}) \subseteq \mathbb{N}$ .

If  $X$  is any object of  $\mathcal{C}$ , then its class  $x$  in  $\mathcal{G}(\mathcal{C})$  decomposes as  $x = \sum_{y \in \text{Irr}(\mathcal{C})} m(y, x)y$ , where  $m(y, x) = \dim \text{Hom}(Y, X)$  is the multiplicity of  $Y$  in  $X$ , if  $Y$  is an object representing the class  $y \in \text{Irr}(\mathcal{C})$ .

For all  $x, y, z \in \mathcal{G}(\mathcal{C})$ , we have:

$$(2.1) \quad m(x, yz) = m(y^*, zx^*) = m(y, xz^*).$$

Let  $x \in \text{Irr}(\mathcal{C})$ . The stabilizer of  $x$  under left multiplication by elements of  $G(\mathcal{C})$  in the Grothendieck ring will be denoted by  $G[x]$ . So that an invertible element  $g \in G(\mathcal{C})$  belongs to  $G[x]$  if and only if  $gx = x$ .

In view of (2.1), for all  $x \in \text{Irr}(\mathcal{C})$ , we have

$$G[x] = \{g \in G(\mathcal{C}) : m(g, xx^*) > 0\} = \{g \in G(\mathcal{C}) : m(g, xx^*) = 1\}.$$

In particular, we have the following relation in  $\mathcal{G}(\mathcal{C})$ :

$$xx^* = \sum_{g \in G[x]} g + \sum_{y \in \text{Irr}(\mathcal{C}), \text{FPdim } y > 1} m(y, xx^*)y.$$

*Remark 2.1.* Notice that an object  $g \in \mathcal{C}$  is invertible if and only if  $\text{FPdim } g = 1$ .

Suppose that  $\mathcal{C}$  is an integral fusion category with  $|\text{c.d.}(\mathcal{C})| = 2$ . That is,  $\text{c.d.}(\mathcal{C}) = \{1, d\}$  for some integer  $d > 1$ . Then  $d$  divides the order of  $G[x]$  for all  $x \in \text{Irr}(\mathcal{C})$  with  $\text{FPdim } x > 1$ . In particular,  $d$  divides the order of  $G(\mathcal{C})$ , and thus  $G(\mathcal{C}) \neq 1$ .

Indeed, if  $x \in \text{Irr}(\mathcal{C})$  with  $\text{FPdim } x = d$ , we have the relation:

$$xx^* = \sum_{g \in G[x]} g + \sum_{y \in \text{Irr}(\mathcal{C}), \text{FPdim } y = d} m(y, xx^*)y.$$

The claim follows by taking Frobenius-Perron dimensions.

**2.3. Semisimple Hopf algebras.** Let  $H$  be a semisimple Hopf algebra over  $k$ . We next recall some of the terminology and conventions from [24] that will be used throughout this paper.

As an algebra,  $H$  is isomorphic to a direct sum of full matrix algebras

$$(2.2) \quad H \simeq k^{(n)} \oplus \bigoplus_{i=1}^r M_{d_i}(k)^{(n_i)},$$

where  $n = |G(H^*)|$ . The Nichols-Zoeller theorem [27] implies that  $n$  divides both  $\dim H$  and  $n_i d_i^2$ , for all  $i = 1, \dots, r$ .

If we have an isomorphism as in (2.2), we shall say that  $H$  is of *type*  $(1, n; d_1, n_1; \dots; d_r, n_r)$  *as an algebra*. If  $H^*$  is of type  $(1, n; d_1, n_1; \dots; d_r, n_r)$  as an algebra, we shall say that  $H$  is of *type*  $(1, n; d_1, n_1; \dots; d_r, n_r)$  *as a coalgebra*.

Let  $V$  be an  $H$ -module. The *character* of  $V$  is the element  $\chi = \chi_V \in H^*$  defined by  $\chi(h) = \text{Tr}_V(h)$ , for all  $h \in H$ . For a character  $\chi$ , its *degree* is the integer  $\deg \chi = \chi(1) = \dim V$ . The character  $\chi_V$  is called irreducible if  $V$  is irreducible.

The set  $\text{Irr}(H)$  of irreducible characters of  $H$  spans a semisimple subalgebra  $R(H)$  of  $H^*$ , called the character algebra of  $H$ . It is isomorphic, under the map  $V \rightarrow \chi_V$ , to the extension of scalars  $k \otimes_{\mathbb{Z}} \mathcal{G}(\text{Rep } H)$  of the Grothendieck ring of the category  $\text{Rep } H$ . In particular, there is an identification  $\text{Irr}(H) \simeq \text{Irr}(\text{Rep } H)$ .

Under this identification, all properties listed in Subsection 2.2 hold true for characters.

In this context, we have  $G(\text{Rep } H) = G(H^*)$ . The stabilizer of  $\chi$  under left multiplication by elements in  $G(H^*)$  will be denoted by  $G[\chi]$ . By the Nichols-Zoeller theorem [27], we have that  $|G[\chi]|$  divides  $(\deg \chi)^2$ .

Following [12, Chapter 12], we shall denote  $\text{c.d.}(H) = \text{c.d.}(\text{Rep } H)$ . So that,

$$\text{c.d.}(H) = \{\deg \chi \mid \chi \in \text{Irr}(H)\}.$$

In particular, if  $H$  is of type  $(1, n; d_1, n_1; \dots; d_r, n_r)$  as an algebra, then  $\text{c.d.}(H) = \{1, d_1, \dots, d_r\}$ .

There is a bijective correspondence between Hopf algebra quotients of  $H$  and standard subalgebras of  $R(H)$ , that is, subalgebras spanned by irreducible characters of  $H$ . This correspondence assigns to the Hopf algebra quotient  $H \rightarrow \overline{H}$  its character algebra  $R(\overline{H}) \subseteq R(H)$ . See [26].

**2.4. Group-theoretical categories.** A group-theoretical fusion category is a fusion category Morita equivalent to a pointed fusion category  $\mathcal{C}(G, \omega)$ . Such a fusion category is equivalent to the category  $\mathcal{C}(G, \omega, F, \alpha)$  of  $k_\alpha F$ -bimodules in  $\mathcal{C}(G, \omega)$ , where  $G$  is a finite group,  $\omega$  is a 3-cocycle on  $G$ ,  $F \subseteq G$  is a subgroup and  $\alpha \in C^2(F, k^\times)$  is a 2-cochain on  $F$  such that  $\omega|_F = d\alpha$ . A semisimple Hopf algebra  $H$  is called group-theoretical if the category  $\text{Rep } H$  is group-theoretical.

Let  $\mathcal{C} = \mathcal{C}(G, \omega, F, \alpha)$  be a group-theoretical fusion category. Let also  $\Gamma$  be a subgroup of  $G$ , endowed with a 2-cocycle  $\beta \in Z^2(\Gamma, k^\times)$ , such that

- The class  $\omega|_\Gamma$  is trivial;
- $G = F\Gamma$ ;
- The class  $\alpha|_{F \cap \Gamma} \beta^{-1}|_{F \cap \Gamma}$  is non-degenerate.

Then there is an associated semisimple Hopf algebra  $H$ , such that the category  $\text{Rep } H$  is equivalent to  $\mathcal{C}$ . By [28], equivalence classes subgroups  $\Gamma$  of

$G$  satisfying the conditions above, classify fiber functors  $\mathcal{C} \mapsto \text{Vec}_k$ ; these correspond to the distinct Hopf algebras  $H$ .

Let  $\mathcal{C} = \mathcal{C}(G, \omega, F, \alpha)$  be a group-theoretical fusion category. The simple objects of  $\mathcal{C}$  are classified by pairs  $(s, U_s)$ , where  $s$  runs over a set of representatives of the double cosets of  $F$  in  $G$ , that is, orbits of the action of  $F$  in the space  $F \backslash G$  of left cosets of  $F$  in  $G$ ,  $F_s = F \cap sFs^{-1}$  is the stabilizer of  $s \in F \backslash G$ , and  $U_s$  is an irreducible representation of the twisted group algebra  $k_{\sigma_s} F_s$ , that is, an irreducible projective representation of  $F_s$  with respect to certain 2-cocycle  $\sigma_s$  determined by  $\omega$ . See [10, Theorem 5.1].

The irreducible representation  $W_{(s, U_s)}$  corresponding to such a pair  $(s, U_s)$  has dimension

$$(2.3) \quad \dim W_{(s, U_s)} = [F : F_s] \dim U_s.$$

As a consequence, we have:

**Corollary 2.2.** *The irreducible degrees of  $\mathcal{C}(G, \omega, F, \alpha)$  divide the order of  $F$ .*

*Remark 2.3.* A group-theoretical category  $\mathcal{C} = \mathcal{C}(G, \omega, F, \alpha)$  is an integral fusion category. An explicit construction of a quasi-Hopf algebra  $H$  such that  $\text{Rep } H \simeq \mathcal{C}$  was given in [21].

As an algebra,  $H$  is a crossed product  $k^{F \backslash G} \#_{\sigma} kF$ , where  $F \backslash G$  is the space of left cosets of  $F$  in  $G$  with the natural action of  $F$ , and  $\sigma$  is a certain 2-cocycle determined by  $\omega$ .

Irreducible representations of  $H$ , that is, simple objects of  $\mathcal{C}$ , can therefore be described using the results for group crossed products in [18]: this is done in [21, Proposition 5.5].

By [10, Theorem 5.2], the group  $G(\mathcal{C})$  of invertible objects of  $\mathcal{C}$  fits into an exact sequence

$$(2.4) \quad 1 \rightarrow \hat{F} \rightarrow G(\mathcal{C}) \rightarrow K \rightarrow 1,$$

where  $K = \{x \in N_G(F) : [\sigma_x] = 1\}$ .

**2.5. Abelian extensions.** Suppose that  $G = F\Gamma$  is an exact factorization of the finite group  $G$ , where  $\Gamma$  and  $F$  are subgroups of  $G$ . Equivalently,  $F$  and  $\Gamma$  form a *matched pair* of groups with the actions  $\triangleleft: \Gamma \times F \rightarrow \Gamma$ ,  $\triangleleft: \Gamma \times F \rightarrow F$ , defined by  $sx = (x \triangleleft s)(x \triangleright s)$ ,  $x \in F$ ,  $s \in \Gamma$ . In this case,  $G$  is isomorphic to the group  $F \bowtie \Gamma$  defined as follows:  $F \bowtie \Gamma = F \times \Gamma$ , with multiplication  $(x, s)(t, y) = (x(s \triangleright y), (s \triangleleft y)t)$ , for all  $x, y \in F$ ,  $s, t \in \Gamma$ .

Let  $\sigma \in Z^2(F, (k^{\Gamma})^{\times})$  and  $\tau \in Z^2(\Gamma, (k^F)^{\times})$  be normalized 2-cocycles with the respect to the actions afforded, respectively, by  $\triangleleft$  and  $\triangleright$ , subject to appropriate compatibility conditions [16].

The bicrossed product  $H = k^{\Gamma} \tau \#_{\sigma} kF$  associated to this data is a semisimple Hopf algebra. There is an *abelian* exact sequence

$$(2.5) \quad k \rightarrow k^{\Gamma} \rightarrow H \rightarrow kF \rightarrow k.$$



Moreover, every Hopf algebra  $H$  fitting into such an exact sequence can be described in this way. This gives a bijective correspondence between the equivalence classes of Hopf algebra extensions (2.5) associated to the matched pair  $(F, \Gamma)$  and a certain abelian group  $\text{Opext}(k^\Gamma, kF)$ .

*Remark 2.4.* The Hopf algebra  $H$  is group theoretical. In fact, we have an equivalence of fusion categories  $\text{Rep } H \simeq \mathcal{C}(G, \omega, F, 1)$  [20, 4.2], where  $\omega$  is the 3-cocycle on  $G$  coming from the so called *Kac exact sequence*.

Irreducible representations of  $H$  are classified by pairs  $(s, U_s)$ , where  $s$  runs over a set of representatives of the orbits of the action of  $F$  in  $\Gamma$ ,  $F_s = F \cap sFs^{-1}$  is the stabilizer of  $s \in \Gamma$ , and  $U_s$  is an irreducible representation of the twisted group algebra  $k_{\sigma_s} F_s$ , that is, an irreducible projective representation of  $F_s$  with cocycle  $\sigma_s$ , where  $\sigma_s(x, y) = \sigma(x, y)(s)$ ,  $x, y \in F$ ,  $s \in \Gamma$ . See [14].

Note that, for all  $s \in \Gamma$ , the restriction of  $\sigma_s : F \times F \rightarrow k^\times$  to the stabilizer  $F_s$  defines indeed a 2-cocycle on  $F_s$ .

The irreducible representation corresponding to such a pair  $(s, U_s)$  is in this case of the form

$$(2.6) \quad W_{(s, U_s)} := \text{Ind}_{k^\Gamma \otimes kF_s}^H s \otimes U_s.$$

**2.6. Quasitriangular Hopf algebras.** Let  $H$  be a finite dimensional Hopf algebra. Recall that  $H$  is called *quasitriangular* if there exists an invertible element  $R \in H \otimes H$ , called an *R-matrix*, such that

$$\begin{aligned} (\Delta \otimes \text{id})(R) &= R_{13}R_{23}, & (\epsilon \otimes \text{id})(R) &= 1, \\ (\text{id} \otimes \Delta)(R) &= R_{13}R_{12}, & (\text{id} \otimes \epsilon)(R) &= 1, \\ \Delta^{\text{cop}}(h) &= R\Delta(h)R^{-1}, & \forall h \in H. \end{aligned}$$

The existence of an *R-matrix* (also called a *quasitriangular structure* in what follows) amounts to the category  $\text{Rep } H$  being a braided tensor category. See [1].

For instance, the group algebra  $kG$  of a finite group  $G$  is a quasitriangular Hopf algebra with  $R = 1 \otimes 1$ . On the other hand, the dual Hopf algebra  $k^G$  admits a quasitriangular structure if and only if  $G$  is abelian.

If it exists, a quasitriangular structure in a Hopf algebra  $H$  need not be unique.

Another example of a quasitriangular Hopf algebra is the *Drinfeld double*  $D(H)$  of  $H$ , where  $H$  is any finite dimensional Hopf algebra. We have  $D(H) = H^{*\text{cop}} \otimes H$  as coalgebras, with a canonical *R-matrix*  $\mathcal{R} = \sum_i h^i \otimes h_i$ , where  $(h_i)_i$  is a basis of  $H$  and  $(h^i)_i$  is the dual basis.

As braided tensor categories,  $\text{Rep } D(H) = \mathcal{Z}(\text{Rep } H)$  is isomorphic to the center of the tensor category  $\text{Rep } H$ .

Suppose  $(H, R)$  is a quasitriangular Hopf algebra. There are Hopf algebra maps  $f_R : H^{*\text{cop}} \rightarrow H$  and  $f_{R_{21}} : H^* \rightarrow H^{\text{op}}$  defined by

$$f_R(p) = p(R^{(1)})R^{(2)}, \quad f_{R_{21}}(p) = p(R^{(2)})R^{(1)},$$

for all  $p \in H^*$ , where  $R = R^{(1)} \otimes R^{(2)} \in H \otimes H$ .

We shall denote  $f_R(H^*) = H_+$  and  $f_{R_{21}}(H^*) = H_-$ , respectively. So that  $H_+, H_-$  are Hopf subalgebras of  $H$  and we have  $H_+ \simeq (H_-^*)^{\text{cop}}$ .

We shall also denote by  $H_R = H_-H_+ = H_+H_-$  the minimal quasitriangular Hopf subalgebra of  $H$ . See [30].

By [30, Theorem 2], the multiplication of  $H$  determines a surjective Hopf algebra map  $D(H_-) \rightarrow H_R$ .

A quasitriangular Hopf algebra  $(H, R)$  is called *factorizable* if the map  $\Phi_R : H^* \rightarrow H$  is an isomorphism, where

$$(2.7) \quad \Phi_R(p) = p(Q^{(1)})Q^{(2)}, \quad p \in H^*;$$

here,  $Q = Q^{(1)} \otimes Q^{(2)} = R_{21}R \in H \otimes H$  [31].

If on the other hand  $\Phi_R = \epsilon 1$  (or equivalently,  $R_{21}R = 1 \otimes 1$ ), then  $(H, R)$  is called *triangular*. Finite dimensional triangular Hopf algebras were completely classified in [6]. In particular, if  $(H, R)$  is a semisimple quasitriangular Hopf algebra, then  $H$  is isomorphic, as a Hopf algebra, to a twisting  $(kG)^J$  of some finite group  $G$ .

It is well-known that the Drinfeld double  $(D(H), \mathcal{R})$  is indeed a *factorizable* quasitriangular Hopf algebra. We have  $D(H)_+ = H$ ,  $D(H)_- = H^{*\text{cop}}$ .

We shall use later on in this paper the following result about factorizable Hopf algebras. A categorical version is established in [11].

**Theorem 2.5.** [33, Theorem 2.3]. *Let  $(H, R)$  be a factorizable Hopf algebra. Then the map  $\Phi_R$  induces an isomorphism of groups  $G(H^*) \rightarrow G(H) \cap Z(H)$ .*

Note that we may identify  $G(D(H)) = G(H^*) \times G(H)$ . Under this identification, Theorem 2.5 gives us a group isomorphism  $G(D(H)^*) \rightarrow G(D(H)) \cap Z(D(H))$ , such that  $g \# f \mapsto f \# g$ . See also [30].

In particular, if  $f = \epsilon$ , then  $g \in G(H) \cap Z(H)$ , and also if  $g = 1$ , then  $f \in G(H^*) \cap Z(H^*)$ .

Suppose  $(H, R)$  is a finite dimensional quasitriangular Hopf algebra, and let  $D(H)$  be the Drinfeld double of  $H$ . Then there is a surjective Hopf algebra map  $f : D(H) \rightarrow H$ , such that  $(f \otimes f)\mathcal{R} = R$ . The map  $f$  is determined by  $f(p \otimes h) = f_R(p)h$ , for all  $p \in H^*$ ,  $h \in H$ .

This corresponds to the canonical inclusion of the braided tensor category  $\text{Rep } H$  (with braiding determined by the action of the  $R$ -matrix) into its center.

In particular, in the case where  $H$  is quasitriangular, the group  $G(H^*)$  can be identified with a subgroup of  $G(D(H)^*)$ .

## 3. NILPOTENCY

Let  $G$  be a finite group. A  $G$ -grading of a fusion category  $\mathcal{C}$  is a decomposition of  $\mathcal{C}$  as a direct sum of full abelian subcategories  $\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$ , such that  $\mathcal{C}_g^* = \mathcal{C}_{g^{-1}}$  and the tensor product  $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$  maps  $\mathcal{C}_g \times \mathcal{C}_h$  to  $\mathcal{C}_{gh}$ . The neutral component  $\mathcal{C}_e$  is thus a fusion subcategory of  $\mathcal{C}$ .

The grading is called *faithful* if  $\mathcal{C}_g \neq 0$ , for all  $g \in G$ . In this case,  $\mathcal{C}$  is called a  $G$ -extension of  $\mathcal{C}_e$  [8].

The following proposition is a consequence of [11, Theorem 3.8].

**Proposition 3.1.** *Let  $\mathcal{C} = \text{Rep } H$ , where  $H$  is a semisimple Hopf algebra. Then a faithful  $G$ -grading on  $\mathcal{C}$  corresponds to a central exact sequence of Hopf algebras  $k \rightarrow k^G \rightarrow H \rightarrow \overline{H} \rightarrow k$ , such that  $\text{Rep } \overline{H} = \mathcal{C}_e$ .*

Let  $\mathcal{C}$  be a fusion category and let  $\mathcal{C}_{\text{ad}}$  be the adjoint subcategory of  $\mathcal{C}$ . That is,  $\mathcal{C}_{\text{ad}}$  is the fusion subcategory of  $\mathcal{C}$  generated by  $X \otimes X^*$ , where  $X$  runs through the simple objects of  $\mathcal{C}$ .

It is shown in [11] that there is a canonical faithful grading on  $\mathcal{C}$ :  $\mathcal{C} = \bigoplus_{g \in U(\mathcal{C})} \mathcal{C}_g$ , called the *universal grading*, such that  $\mathcal{C}_e = \mathcal{C}_{\text{ad}}$ . The group  $U(\mathcal{C})$  is called the *universal grading group* of  $\mathcal{C}$ .

In the case where  $\mathcal{C} = \text{Rep } H$ , for a semisimple Hopf algebra  $H$ ,  $K = k^{U(\mathcal{C})}$  is the maximal central Hopf subalgebra of  $H$  and  $\mathcal{C}_{\text{ad}} = \text{Rep } H/HK^+$  [11, Theorem 3.8].

Recall from [11, 8] that a fusion category  $\mathcal{C}$  is called (cyclically) *nilpotent* if there is a sequence of fusion categories

$$\mathcal{C}_0 = \text{Vec}, \mathcal{C}_1, \dots, \mathcal{C}_n = \mathcal{C}$$

and a sequence  $G_1, \dots, G_n$  of finite (cyclic) groups such that  $\mathcal{C}_i$  is faithfully graded by  $G_i$  with trivial component  $\mathcal{C}_{i-1}$ .

The semisimple Hopf algebra  $H$  is called nilpotent if the fusion category  $\text{Rep } H$  is nilpotent [11, Definition 4.4].

For instance, if  $G$  is a finite group, then the dual group algebra  $k^G$  is always nilpotent. However, the group algebra  $kG$  is nilpotent if and only if the group  $G$  is nilpotent [11, Remark 4.7. (1)].

**3.1. Nilpotency of an abelian extension.** It is shown in [10, Corollary 4.3] that a group-theoretical fusion category  $\mathcal{C}(G, \omega, F, \alpha)$  is nilpotent if and only if the normal closure of  $F$  in  $G$  is nilpotent. On the other hand, this happens if and only if  $F$  is nilpotent and subnormal in  $G$ , if and only if  $F \subseteq \text{Fit}(G)$ , where  $\text{Fit}(G)$  is the Fitting subgroup of  $G$ , that is, the unique largest normal nilpotent subgroup of  $G$  [10, Subsection 2.3].

Combined with Remark 2.4, this implies:

**Proposition 3.2.** *Let  $k \rightarrow k^\Gamma \rightarrow H \rightarrow kF \rightarrow k$  be an abelian exact sequence and let  $G = F \bowtie \Gamma$  be the associated factorizable group. Then  $H$  is nilpotent if and only if  $F \subseteq \text{Fit}(G)$ .*

An abelian exact sequence (2.5) is called *central* if the image of  $k^\Gamma$  is a central Hopf subalgebra of  $H$ . It is called *cocentral*, if the dual exact sequence is central.

The following facts are well-known:

**Lemma 3.3.** *Consider an abelian exact sequence (2.5). Then we have:*

- (i) *The sequence is central if and only if the action  $\triangleleft: \Gamma \times F \rightarrow \Gamma$  is trivial. In this case, the group  $G = F \rtimes \Gamma$  is a semidirect product  $G \simeq F \rtimes \Gamma$  with respect to the action  $\triangleright: \Gamma \times F \rightarrow F$ .*
- (ii) *The sequence is cocentral if and only if the action  $\triangleright: \Gamma \times F \rightarrow F$  is trivial. In this case, the group  $G = F \rtimes \Gamma$  is a semidirect product  $G \simeq F \rtimes \Gamma$  with respect to the action  $\triangleleft: \Gamma \times F \rightarrow \Gamma$ .  $\square$*

*Remark 3.4.* Assume the exact sequence (2.5) is central. Then  $F$  is a normal subgroup of  $G$ . It follows from Proposition 3.2 that  $H$  is nilpotent if and only if  $F$  is nilpotent.

#### 4. ON THE NILPOTENCY OF A CLASS OF SEMISIMPLE HOPF ALGEBRAS

Let  $p$  be a prime number. We shall consider in this subsection a nontrivial semisimple Hopf algebra  $H$  fitting into an abelian exact sequence

$$(4.1) \quad k \rightarrow k^{\mathbb{Z}_p} \rightarrow H \rightarrow kF \rightarrow k.$$

The main result of this subsection is Proposition 4.3 below.

We first have the following lemma.

Suppose that  $\mathcal{C}$  is any group-theoretical fusion category of the form  $\mathcal{C} = \mathcal{C}(G, \omega, \mathbb{Z}_p, \alpha)$  (Note that we may assume that  $\alpha = 1$ .) In particular,  $p$  divides the order of  $G(\mathcal{C})$ . We also have  $\text{c.d.}(\mathcal{C}) \subseteq \{1, p\}$ , by Corollary 2.2.

**Lemma 4.1.** *Let  $\mathcal{C} = \mathcal{C}(G, \omega, \mathbb{Z}_p, \alpha)$ . Assume that  $|G(\mathcal{C})| = p$ . Then  $G$  is a Frobenius group with Frobenius complement  $\mathbb{Z}_p$ .*

*Proof.* The description of the irreducible representations of  $\mathcal{C}$  in Subsection 2.4, combined with the assumption that  $|G(\mathcal{C})| = p$ , implies that  $g\mathbb{Z}_p g^{-1} \cap \mathbb{Z}_p = \{e\}$ , for all  $g \in G \setminus \mathbb{Z}_p$ . (In particular, the action of  $\mathbb{Z}_p$  on  $\mathbb{Z}_p \setminus G$  has no fixed points  $s \neq e$ .)

This condition says that  $G$  is a Frobenius group with Frobenius complement  $\mathbb{Z}_p$ , as claimed.  $\square$

*Remark 4.2.* Let  $G$  be a Frobenius group with Frobenius complement  $\mathbb{Z}_p$ , as in Lemma 4.1. By Frobenius' Theorem we have that the Frobenius kernel  $N$  is a normal subgroup of  $G$ , such that  $G$  is a semidirect product  $G = N \rtimes \mathbb{Z}_p$ . Moreover,  $N$  is a nilpotent group, by a theorem of Thompson. See [32, Theorem 10.5.6], [12, Theorem (7.2)]. In fact, the Frobenius kernel  $N$  is equal to  $\text{Fit}(G)$ , the Fitting subgroup of  $G$  [32, Exercise 10.5.8].

As a consequence we get the following:

**Proposition 4.3.** *Consider the abelian exact sequence (4.1) and assume that  $|G(H)| = p$ . Then we have:*

- (i) *The sequence is central, that is,  $G(H) \subseteq Z(H)$ .*
- (ii)  *$G = F \rtimes \mathbb{Z}_p$  is a Frobenius group with kernel  $F$ . In particular,  $F$  is nilpotent.*

*Proof.* We follow the lines of the proof of [13, Proposition X.7 (i)]. Consider the matched pair  $(F, \mathbb{Z}_p)$  associated to (4.1), as in Subsection (2.5). Let  $G = F \rtimes \mathbb{Z}_p$  be the corresponding factorizable group.

We have an equivalence of fusion categories  $\text{Rep } H^* \simeq \mathcal{C}(G, \omega, \mathbb{Z}_p, 1)$ , see Remark 2.4. Then  $\text{Rep } H^*$  is group-theoretical and by assumption,  $G(\text{Rep } H^*)$  is of order  $p$ . By Lemma 4.1,  $G$  is a Frobenius group with Frobenius complement  $\mathbb{Z}_p$ . Therefore  $G$  is a semidirect product  $G = N \rtimes \mathbb{Z}_p$ , where  $N = \text{Fit}(G)$  is a nilpotent subgroup (see Remark 4.2).

Since  $|G(H)| = p$ , then the action of  $\mathbb{Z}_p$  on  $F$  has no fixed points. It follows, after decomposing  $F$  as a disjoint union of  $\mathbb{Z}_p$ -orbits, that  $|F| = 1 \pmod{p}$ . In particular,  $|F|$  is not divisible by  $p$ . Then  $F$  must map trivially under the canonical projection  $G \rightarrow G/N$ , that is,  $F \subseteq N$ . Hence  $F = N$ , because they have the same order. This shows (ii). Since  $F$  is normal in  $G$ , we get (i) in view of Lemma 3.3.  $\square$

**Corollary 4.4.** *Let  $k \rightarrow k^{\mathbb{Z}_p} \rightarrow H \rightarrow kF \rightarrow k$  be an abelian exact sequence such that  $|G(H)| = p$ . Then  $H$  is nilpotent.*

*Proof.* It follows from Proposition 4.3, in view of Remark 3.4.  $\square$

*Remark 4.5.* In view of [13, Theorem IX.8 (iii)], if  $H$  is a Kac algebra with  $\text{c.d.}(H^*) = \{1, p\}$  and  $|G(H)| = p$ , then  $H$  is a central abelian extension associated to an action of the cyclic group of order  $p$  on a nilpotent group. It follows from Corollary 4.4 that  $H$  is a nilpotent Hopf algebra.

*Remark 4.6.* Note that the (dual) assumption that  $\text{c.d.}(H) = \{1, p\}$  does not imply that  $H$  is nilpotent in general. For example, take  $H$  to be the group algebra of a nonabelian semidirect product  $F \rtimes \mathbb{Z}_p$ , where  $F$  is an abelian group such that  $(|F|, p) = 1$ .

On the other hand, the assumption on  $|G(H)|$  in Corollary 4.4 and Proposition 4.3 is essential. Namely, for all prime number  $p$ , there exist semisimple Hopf algebras  $H$  with  $\text{c.d.}(H^*) = \{1, p\}$  and such that  $H$  is *not* nilpotent.

To see an example, consider a group  $F$  with an automorphism of order  $p$  and suppose  $F$  is not nilpotent (take for instance  $F = \mathbb{S}_n$ , a symmetric group, such that  $n > 6$  is sufficiently large). Consider the corresponding action of  $\mathbb{Z}_p$  on  $F$  by group automorphisms and let  $G = F \rtimes \mathbb{Z}_p$  be the semidirect product.

Then there is an associated (split) abelian exact sequence  $k \rightarrow k^{\mathbb{Z}_p} \rightarrow H \rightarrow kF \rightarrow k$ , such that  $H$  is not commutative and not cocommutative. Moreover, in view of Remark 2.2,  $\text{c.d.}(H^*) = \{1, p\}$ . But, by Remark 3.4,  $H$  is not nilpotent, because  $F$  is not nilpotent by assumption.

**4.1. Reduction to abelian extensions from character degrees.** In this subsection we consider the case where  $\text{c.d.}(H) = \{1, p\}$  for some prime  $p$  and  $|G(H^*)| = p$ . We treat the problem of deducing an abelian extension like (4.1) from this assumption.

It is known, for instance, that if  $p = 2$ , then the assumption implies that  $H$  is cocommutative [2, Proposition 6.8], [13, Corollary IX.9].

**Lemma 4.7.** *Suppose that  $\text{c.d.}(H^*) = \{1, p\}$  for some prime  $p$ . Then  $H/(kG(H))^+H$  is a cocommutative coalgebra.*

*Proof.* Let  $\chi$  be an irreducible character of degree  $p$ . We have that

$$\chi\chi^* = \sum_{g \in G[\chi]} g + \sum_{\deg \lambda = p} \lambda.$$

So  $p \mid |G[\chi]|$ . Therefore  $|G[\chi]|$  is either  $p = \deg \chi$  or  $p^2$ , because it divides  $(\deg \chi)^2$ .

Moreover, since  $\chi = g\chi$  for all  $g \in G[\chi]$ , we have  $G[\chi]C = C$ , where  $C$  is the simple subcoalgebra of  $H$  containing  $\chi$ . Then it follows from [24, Remark 3.2.7] that  $C/(kG[\chi])^+C$  is a cocommutative coalgebra (indeed,  $|G[\chi]|$  is either  $p = \deg \chi$  or  $p^2$ , but in the last case,  $C/(kG[\chi])^+C$  is one-dimensional, hence also cocommutative). Then  $H/(kG(H))^+H$  is a cocommutative coalgebra, by [24, Corollary 3.3.2].  $\square$

**4.2. Results for the type  $(1, p; p, n)$ .** Let  $p$  be a prime number. Along this subsection  $H$  will be a semisimple Hopf algebra such that  $\text{c.d.}(H) = \{1, p\}$  and  $|G(H^*)| = p$ . So that  $H$  is of type  $(1, p; p, n)$  as an algebra.

**Proposition 4.8.** *Suppose that  $p$  divides  $|G(H)|$ . Then  $G(H^*) \subseteq Z(H^*)$  and  $H^*$  is nilpotent.*

*Proof.* By assumption, there is a subgroup  $G$  of  $G(H)$  with  $|G| = p$  (i.e.  $G \simeq \mathbb{Z}_p$ ) and the Hopf algebra inclusion  $kG \rightarrow H$  induces the following sequence:

$$kG(H^*) \xrightarrow{i} H^* \xrightarrow{\pi} kG,$$

with  $\pi$  surjective. By [24, Lemma 4.1.9], setting  $A = kG(H^*)$  and  $B = kG$ , we have that  $\pi \circ i : kG(H^*) \rightarrow kG$  is an isomorphism and  $H^* \simeq R \# kG(H^*) \simeq R \# \mathbb{Z}_p$  is a biproduct, where  $R \doteq (H^*)^{\text{co} \pi}$  is a semisimple braided Hopf algebra over  $\mathbb{Z}_p$ . The coalgebra  $R$  is cocommutative, by Lemma (4.7), because  $R \simeq H^*/H^*kG(H^*)^+$  as coalgebras. Since  $p \nmid 1 + np = \dim R$  then by [36, Proposition 7.2],  $R$  is trivial. Therefore, by [24, Proposition 4.6.1],  $H^*$  fits into an abelian *central* exact sequence

$$k \rightarrow k\mathbb{Z}_p \rightarrow H^* \rightarrow R \rightarrow k.$$

Now, since the extension is abelian, there is a group  $F$  such that  $R \simeq kF$ . It follows from Corollary 4.4 that  $H^*$  is nilpotent.  $\square$

**Proposition 4.9.** *Suppose  $H$  is quasitriangular. Then  $G(H^*) \subseteq Z(H^*)$  and  $H^*$  is nilpotent.*

*Proof.* Consider the Drinfeld double  $D(H)$ . Since  $H$  is quasitriangular,  $G(H^*) \simeq \mathbb{Z}_p$  is isomorphic to a subgroup of  $G(D(H)^*)$ . Then  $G(D(H)^*)$  has an element  $g\#f$  of order  $p$ . We have that  $G(D(H)^*) \simeq G(D(H)) \cap Z(D(H)) \subseteq G(D(H)) = G(H^*) \times G(H)$ ; see Subsection 2.6.

In particular, the element  $f\#g \in G(D(H)) \cap Z(D(H))$  is of order  $p$ . If  $g$  is of order  $p$ , then the proposition follows from Proposition 4.8. Thus we may assume that  $g = 1$ . Then  $f \in G(H^*) \cap Z(H^*)$  is of order  $p$ , implying that  $G(H^*) \subseteq Z(H^*)$ .

Therefore  $H^*$  fits into an abelian central exact sequence  $k \rightarrow k^{\mathbb{Z}_p} \rightarrow H^* \rightarrow kF \rightarrow k$ , where  $F$  is a finite group such that  $kF \simeq H^*/H^*(k^{\mathbb{Z}_p})^+$ , by Lemma 4.7. In view of the assumption on the algebra structure of  $H$ , Corollary 4.4 implies that  $H^*$  is nilpotent, as claimed.  $\square$

**4.3. Results for the type  $(1, p; p, 1)$ .** We next discuss the case where  $H$  is of type  $(1, p; p, 1)$  as an algebra (not necessarily quasitriangular). In particular,  $\dim H = p(p+1)$  is even.

Notice that under this assumption, the category  $\text{Rep } H$  is a *near-group category* with fusion rule given by the group  $G = G(H^*) \simeq \mathbb{Z}_p$  and the integer  $\kappa$  [35].

Let  $\chi$  be the irreducible character of degree  $p$ . It follows that  $\chi = \chi^*$  and  $\chi g = \chi = g\chi$ . Then

$$\chi^2 = \sum_{g \in G(H^*)} g + \kappa\chi.$$

Taking degrees in the equation above we obtain  $p^2 = p + \kappa p$ , which means that  $\kappa = p - 1$ .

We shall use the following proposition. A more general statement will be proved in Theorem 6.2.

**Proposition 4.10.** *Suppose  $H$  is of type  $(1, p; p, 1)$  as an algebra. Then one of the following holds:*

- (i)  $p = 2$  and  $H \simeq kS_3$ , or
- (ii)  $p = 2^\alpha - 1$ <sup>1</sup>, and  $\dim H = 2^\alpha p$ .

*In particular,  $H$  is solvable.*

*Proof.* By [35, Theorem 1.2], it follows that  $G(H^*) \simeq \mathbb{Z}_{q^\alpha - 1}$ , for some prime  $q$  and  $\alpha \geq 1$ . Therefore  $p = q^\alpha - 1$ . If  $q > 2$ , then  $p = 2$ , which implies  $H \simeq kS_3$  is cocommutative. If  $q = 2$ , then  $p$  has the particular expression  $p = 2^\alpha - 1$ .

Hence  $\dim H$  equals 6 or  $p(p+1) = 2^\alpha p$ . By Burnside's theorem for fusion categories [8, Theorem 1.6],  $H$  is solvable.  $\square$

*Remark 4.11.* Let  $p$  be a prime number such that  $p = 2^\alpha - 1$ , as in Proposition 4.10. Consider the affine group  $N$  of the field  $\mathbb{F}_{2^\alpha}$ , that is,  $N$  is the semidirect product  $\mathbb{F}_{2^\alpha} \rtimes \mathbb{F}_{2^\alpha}^\times$  with respect to the natural action of  $\mathbb{F}_{2^\alpha}^\times$  on

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<sup>1</sup>Such a prime number is called a *Mersenne prime*, in particular  $\alpha$  must be prime.

$\mathbb{F}_{2^\alpha}$ . Then the group  $N$  has the prescribed algebra type (see [35, Subsection 4.1]).

Furthermore, suppose  $p$  is (any) prime number, and  $N$  is a group whose group algebra has algebra type  $(1, p; p, 1)$ . Then  $N$  has order  $p(p+1)$  and it follows from the main result of [34] that either  $p = 2$  and  $N \simeq \mathbb{S}_3$  or  $p = 2^\alpha - 1$ ,  $\alpha > 1$ , and  $N \simeq \mathbb{F}_{2^\alpha} \rtimes \mathbb{F}_{2^\alpha}^\times$ .

**Proposition 4.12.** *Let  $H$  be a semisimple Hopf algebra of type  $(1, p; p, 1)$  as an algebra. Then  $G(H^*) \subseteq Z(H^*)$  and  $H^*$  is nilpotent.*

*Proof.* We have just proved in Proposition 4.10 that under this hypothesis  $H$  is solvable. Since  $\text{Rep } D(H) \simeq Z(\text{Rep } H)$ , then  $D(H)$  is also solvable [8, Proposition 4.5 (i)].

By [8, Proposition 4.5 (iv)],  $D(H)$  has nontrivial representations of dimension 1, that is,  $|G(D(H)^*)| \neq 1$ . We have that  $G(D(H)^*) \simeq G(D(H)) \cap Z(D(H)) \subseteq G(D(H)) = G(H^*) \times G(H)$ ; see Subsection 2.6.

We next argue as in the proof of Proposition 4.9. Consider an element  $1 \neq f \# g \in G(D(H)) \cap Z(D(H))$ . If  $f = 1$ , then  $1 \neq g \in Z(H) \cap G(H)$ . Therefore,  $H^*$  fits into a cocentral extension  $k \rightarrow K \rightarrow H^* \rightarrow k^{(g)} \rightarrow k$ , where  $K$  is a *proper* normal Hopf subalgebra. The assumption on the algebra structure of  $H$  implies that  $K = kG(H^*)$ . Thus  $kG(H^*)$  is normal in  $H^*$ , and the extension is abelian, by Lemma 4.7. The proposition follows in this case from Proposition 4.3 (i) and Corollary 4.4.

Thus we may assume that  $f \neq 1$ . In particular,  $f$  has order  $p$ .

If  $|f| = |g| = p = |G(H^*)|$ , we have that  $p \mid |G(H)|$ . Then  $G(H^*) \subseteq Z(H^*)$  and  $H^*$  is nilpotent, by Proposition 4.8.

Otherwise, take  $|g| = n$ , with  $p \neq n$ . If  $f^n = 1$ , then  $p$  divides  $n$  and thus  $p$  divides  $|G(H)|$ . As before, we are done by Proposition 4.8.

If  $f^n \neq 1$ , then  $f^n \# 1 = (f^n \# g^n) = (f \# g)^n \in Z(D(H))$ , which implies that  $f^n \neq 1$  is central in  $H^*$  and thus  $G(H^*) \subseteq Z(H^*)$ .

Therefore  $H^*$  fits into an abelian central exact sequence  $k \rightarrow k^{\mathbb{Z}_p} \rightarrow H^* \rightarrow kF \rightarrow k$ , where  $F$  is a finite group such that  $kF \simeq H^*/H^*(k^{\mathbb{Z}_p})^+$ , by Lemma 4.7. In view of the assumption on the algebra structure of  $H$ , Corollary 4.4 implies that  $H^*$  is nilpotent, as claimed.  $\square$

**Theorem 4.13.** *Let  $H$  be a semisimple Hopf algebra of type  $(1, p, p, 1)$  as an algebra. Then either  $p = 2$  and  $H \simeq k\mathbb{S}_3$ , or  $H$  is isomorphic to a twisting of the group algebra  $kN$ , where  $p = 2^\alpha - 1$ ,  $\alpha > 1$ , and  $N$  is the affine group of the field  $\mathbb{F}_{2^\alpha}$ .*

*Proof.* If  $p = 2$ , then  $\dim H = 6$  and the result follows from [15]. So suppose that  $p$  is odd. By Propositions 4.12 and 4.10,  $H^*$  fits into an abelian central exact sequence  $k \rightarrow k^{\mathbb{Z}_p} \rightarrow H^* \rightarrow kF \rightarrow k$ , where  $F$  is a finite group of order  $p+1 = 2^\alpha$ . Then the action  $\triangleleft: \mathbb{Z}_p \times F \rightarrow \mathbb{Z}_p$  is trivial, while the action  $\triangleright: \mathbb{Z}_p \times F \rightarrow F$  is determined by an automorphism  $\varphi \in \text{Aut } F$  of order  $p = 2^\alpha - 1$ .



We first claim that the group  $F$  must be abelian. By a result of P. Hall [32, 5.3.3], since  $F$  is a 2-group, the order of  $\text{Aut } F$  divides the number  $n2^{(\alpha-r)r}$ , where  $n = |\text{GL}(r, 2)|$  and  $2^r$  equals the index in  $F$  of the Frattini subgroup  $\text{Frat}(F)$  (which is defined as the intersection of all the maximal subgroups of  $F$  [32, pp. 135]). In particular, we have  $r \leq \alpha$ .

Since the order of  $\varphi$  divides the order of  $\text{Aut } F$  and  $|\text{GL}(r, 2)| = (2^r - 1)(2^r - 2) \dots (2^r - 2^{r-1})$ , it follows that the prime  $p = 2^\alpha - 1$  divides  $2^r - 1$ , which means that  $r = \alpha$  and, therefore,  $\text{Frat}(F) = 1$ .

Since  $F$  is nilpotent (because it is a 2-group), a result of Wielandt [32, 5.2.16] implies that  $[F, F]$ , the commutator subgroup of  $F$ , is a subgroup of the Frattini subgroup  $\text{Frat}(F)$ . As we have just shown, we have  $\text{Frat}(F) = 1$  in this case. Thus  $[F, F] = 1$  and therefore  $F$  is abelian, as claimed.

Consider the split extension  $B_0 = k^{\mathbb{Z}_p} \# kF$  associated to the matched pair  $(\mathbb{Z}_p, F)$ . Since  $F$  is abelian,  $B_0$  (being a central extension) is commutative. This means that  $B_0$  is isomorphic to  $k^N$ , where  $N = F \rtimes \mathbb{Z}_p$ .

Notice that  $|F| = 2^\alpha$  is relatively prime to  $p$ . It follows from [23, Proposition 5.22] and [17, Proposition 3.1] that  $H^*$  is obtained from the split extension  $B_0 = k^{\mathbb{Z}_p} \# kF \simeq k^N$  by twisting the multiplication. Indeed, the element representing the class of  $H^*$  in the group  $\text{Opext}(kF, k^{\mathbb{Z}_p})$  is the image of an element of  $H^2(F, k^\times)$  under the map  $H^2(F, k^\times) \oplus H^2(\mathbb{Z}_p, k^\times) \simeq H^2(F, k^\times) \rightarrow \text{Opext}(kF, k^{\mathbb{Z}_p})$  in the Kac exact sequence [17, Theorem 1.10]. Then the claim follows from [17, Proposition 3.1]. Dualizing, we get that  $H$  is a twisting of the group algebra of the group  $N$ .

Finally, the assumption on the algebra structure of  $H$  implies that  $N$  is one of the claimed groups. See Remark 4.11.  $\square$

**Corollary 4.14.** *Let  $H$  be a semisimple Hopf algebra of type  $(1, p, p, 1)$  as an algebra. Then  $\text{Rep } H \simeq \text{Rep } N$ , where  $N = \mathbb{S}_3$  or  $N$  is the affine group of the field  $\mathbb{F}_{2^\alpha}$ , for some  $\alpha > 1$ .*

## 5. SOLVABILITY

Recall from [8] that a fusion category  $\mathcal{C}$  is called *weakly group-theoretical* if it is Morita equivalent to a nilpotent fusion category. If, furthermore,  $\mathcal{C}$  is Morita equivalent to a cyclically nilpotent fusion category, then  $\mathcal{C}$  is called *solvable*.

In other words,  $\mathcal{C}$  is weakly group-theoretical (solvable) if there exists an indecomposable algebra  $A$  in  $\mathcal{C}$  such that the category  ${}_A\mathcal{C}_A$  of  $A$ -bimodules in  $\mathcal{C}$  is a (cyclically) nilpotent fusion category.

Note that a group-theoretical fusion category is weakly group-theoretical.

On the other hand, the condition on  $\mathcal{C}$  being solvable is equivalent to the existence of a sequence of fusion categories

$$\mathcal{C}_0 = \text{Vec}_k, \mathcal{C}_1, \dots, \mathcal{C}_n = \mathcal{C},$$

such that  $\mathcal{C}_i$  is obtained from  $\mathcal{C}_{i-1}$  either by a  $G_i$ -equivariantization or as a  $G_i$ -extension, where  $G_1, \dots, G_n$  are cyclic groups of prime order. See [8, Proposition 4.4].

If  $G$  is a finite group and  $\omega \in H^3(G, k^\times)$ , we have that the categories  $\mathcal{C}(G, \omega)$  and  $\text{Rep } G$  are solvable if and only if  $G$  is solvable.

Let us call a semisimple Hopf algebra  $H$  *weakly group-theoretical* or *solvable*, if the category  $\text{Rep } H$  is weakly group-theoretical or solvable, respectively.

**5.1. Solvability of an abelian extension.** By [8, Proposition 4.5 (i)], solvability of a fusion category is preserved under Morita equivalence. Therefore, a group-theoretical fusion category  $\mathcal{C}(G, \omega, F, \alpha)$  is solvable if and only if the group  $G$  is solvable.

*Remark 5.1.* As a consequence of the Feit-Thompson theorem [9], we get that if the order of  $G$  is odd, then  $\mathcal{C}(G, \omega, F, \alpha)$  is solvable. This fact generalizes to weakly group-theoretical fusion categories; see Proposition 7.1 below.

This implies the following characterization of the solvability of an abelian extension:

**Corollary 5.2.** *Let  $H$  be a semisimple Hopf algebra fitting into an abelian exact sequence (2.5), then  $H$  is solvable if and only if  $G = F \rtimes \Gamma$  is solvable.*

In particular, if  $H$  is solvable, then  $F$  and  $\Gamma$  are solvable.

A result of Wielandt [38] implies that if the groups  $\Gamma$  and  $F$  are nilpotent, then  $G$  is solvable. As a consequence, we get the following:

**Corollary 5.3.** *Suppose  $\Gamma$  and  $F$  are nilpotent. Then  $H$  is solvable.*

Then, for instance, the abelian extensions in Proposition 4.3 are solvable.

Combining Corollary 5.3 with Lemma 4.1 and Remark 4.2, we get:

**Corollary 5.4.** *Let  $\mathcal{C} = \mathcal{C}(G, \omega, \mathbb{Z}_p, \alpha)$ . Assume that  $|G(\mathcal{C})| = p$ . Then  $\mathcal{C}$  is solvable.*

## 6. SOLVABILITY FROM CHARACTER DEGREES

Let  $p$  be a prime number. We study in this section fusion categories  $\mathcal{C}$  such that  $\text{c.d.}(\mathcal{C}) = \{1, p\}$ .

It is known that if  $G$  is a finite group, then this assumption implies that the group  $G$ , and thus the category  $\text{Rep } G$ , are solvable [12].

*Remark 6.1.* If  $H$  is any semisimple Hopf algebra such that  $\text{c.d.}(H) = \{1, p\}$  and  $G$  is any finite group, then the tensor product Hopf algebra  $A = H \otimes k^G$  also satisfies that  $\text{c.d.}(A) = \{1, p\}$  (since the irreducible modules of  $A$  are tensor products of irreducible modules of  $H$  and  $k^G$ ).

But  $A$  is not solvable unless  $G$  is solvable; indeed,  $k^G$  is a Hopf subalgebra as well as a quotient Hopf algebra of  $A$ .

Our aim in this section is to prove some structural results on  $\mathcal{C}$ , regarding solvability, under additional restrictions.

The following theorem generalizes Proposition 4.10.

**Theorem 6.2.** *Let  $\mathcal{C}$  be a near-group fusion category such that  $\text{c.d.}(\mathcal{C}) = \{1, p\}$ . Then  $\mathcal{C}$  is solvable.*

*Proof.* In the notation of [35], let the fusion rules of  $\mathcal{C}$  be given by the pair  $(G, \kappa)$ , where  $G$  is the group of invertible objects of  $\mathcal{C}$  and  $\kappa$  is a nonnegative integer. Then  $\text{Irr}(\mathcal{C}) = G \cup \{m\}$ , with the relation

$$(6.1) \quad m^2 = \sum_{g \in G} g + \kappa m.$$

The assumption on  $\text{c.d.}(\mathcal{C})$  implies that  $\text{FPdim } m = p$ . Hence  $\text{FPdim } \mathcal{C} = |G| + p^2$ , and since  $|G| = |G(\mathcal{C})|$  divides  $\text{FPdim } \mathcal{C}$ , we get that  $|G| = p$  or  $p^2$ . (Note that taking Frobenius-Perron dimensions in (6.1), we get that  $G \neq 1$ .)

If  $|G| = p^2$ , then  $\kappa = 0$  and  $\mathcal{C}$  is a Tambara-Yamagami category [37]. Furthermore,  $\mathcal{C}$  is a  $\mathbb{Z}_2$ -extension of a pointed category  $\mathcal{C}(G, \omega)$ . Then  $\mathcal{C}$  is solvable in this case, by [8, Proposition 4.5 (i)].

Suppose that  $|G| = p$ . Then  $\kappa = p - 1$ . As in the proof of Proposition 4.10, using [35, Theorem 1.2], we get that  $\text{FPdim } \mathcal{C} = p(p + 1)$  equals 6 or  $p2^\alpha$ . Then  $\mathcal{C}$  is solvable, by [8, Theorem 1.6].  $\square$

Our next result is the following theorem, for  $\mathcal{C} = \text{Rep } H$ , which is a consequence of Proposition 4.9. A stronger version of this result will be given in Subsection 7.2, under additional dimension restrictions.

**Theorem 6.3.** *Suppose  $H$  is of type  $(1, p; p, n)$  as an algebra. Assume in addition that  $H$  is quasitriangular. Then  $H$  is solvable.*

*Proof.* We have shown in Proposition 4.9 that  $H^*$  is nilpotent. Moreover, by Lemma 4.7,  $H$  fits into an abelian cocentral exact sequence  $k \rightarrow k^F \rightarrow H \rightarrow k\mathbb{Z}_p \rightarrow k$ , where  $F$  is a nilpotent group. Therefore,  $H$  is solvable, by Corollary 5.3.  $\square$

In the remaining of this section, we restrict ourselves to the case where  $\mathcal{C} = \text{Rep } H$  for a semisimple Hopf algebra  $H$ .

**6.1. The case  $p = 2$ .** Let  $H$  be a semisimple Hopf algebra such that  $\text{c.d.}(H) \subseteq \{1, 2\}$ . By [2, Theorem 6.4], one of the following possibilities holds:

- (i) There is a cocentral abelian exact sequence  $k \rightarrow k^F \rightarrow H \rightarrow k\Gamma \rightarrow k$ , where  $F$  is a finite group and  $\Gamma \simeq \mathbb{Z}_2^n$ ,  $n \geq 1$ , or
- (ii) There is a central exact sequence  $k \rightarrow k^U \rightarrow H \rightarrow B \rightarrow k$ , where  $B = H_{\text{ad}}$  is a proper Hopf algebra quotient, and  $U = U(\text{Rep } H)$  is the universal grading group of the category of finite dimensional  $H$ -modules.

In particular, if  $H = H_{\text{ad}}$ , then  $H$  satisfies (i).

As a consequence of this result we have:

**Theorem 6.4.** *Let  $H$  be a semisimple Hopf algebra such that  $\text{c.d.}(H) \subseteq \{1, 2\}$ . Then  $H$  is weakly group-theoretical.*

*Moreover, if  $H = H_{\text{ad}}$ , then  $H$  is group-theoretical.*

*Proof.* The assumption implies that  $H$  satisfies (i) or (ii) above. If  $H$  satisfies (i), then  $H$  is group-theoretical, by Remark 2.4.

Otherwise,  $H$  satisfies (ii), and then the category  $\text{Rep } H$  is a  $U$ -extension of  $\text{Rep } B$ , in view of Proposition 3.1. By an inductive argument, we may assume that  $B$  is weakly group-theoretical (note that  $\text{c.d.}(B) \subseteq \{1, 2\}$ ). Therefore so is  $H$ , by [8, Proposition 4.1].  $\square$

We next discuss conditions that guarantee the solvability of  $H$ . The following result is proved in [2].

**Proposition 6.5.** [2, Proposition 6.8]. *Suppose  $H$  is of type  $(1, 2; 2, n)$  as an algebra. Then  $H$  is cocommutative.*

The proposition implies that such a Hopf algebra  $H$  is isomorphic to a group algebra  $kG$  for some finite group  $G$ . By the assumption on the algebra structure of  $H$ , the group  $G$ , and then also  $H$ , are solvable.

The next lemma gives a sufficient condition for  $H$  to be solvable.

**Lemma 6.6.** *Suppose  $\text{c.d.}(H) \subseteq \{1, 2\}$  and  $H = H_{\text{ad}}$ . Then  $H$  is solvable if and only if the group  $F$  in (i) is solvable.*

*Proof.* Since  $H = H_{\text{ad}}$ , then  $H$  satisfies (i). Therefore  $H$  is solvable if and only if the relevant factorizable group  $G = F \rtimes \Gamma$  is solvable, by Corollary 5.2. Also, since the sequence (i) is cocentral, then  $G$  is a semidirect product:  $G = F \rtimes \Gamma$ . This proves the lemma.  $\square$

*Remark 6.7.* Suppose that  $H$  has a faithful irreducible character  $\chi$  of degree 2, such that  $\chi\chi^* = \chi^*\chi$ . Then it follows from [2, Theorem 3.5] that  $H$  fits into a central abelian exact sequence  $k \rightarrow k^{\mathbb{Z}^m} \rightarrow H \rightarrow kT \rightarrow k$ , for some polyhedral group  $T$  of even order and for some  $m \geq 1$ . In particular, since  $\text{c.d.}(H) = \{1, 2\}$ , then  $T$  is necessarily cyclic or dihedral (see, for instance, [2, pp. 10] for a description of the polyhedral groups and their character degrees). Therefore  $H$  is solvable in this case.

The assumption on  $\chi$  is satisfied in the case where  $H$  is quasitriangular; so that the conclusion holds in this case. We shall show in the next subsection that every quasitriangular semisimple Hopf algebra with  $\text{c.d.}(H) \subseteq \{1, 2\}$  is also solvable.

We next prove some lemmas that will be useful in the next subsection.

**Lemma 6.8.** *Suppose  $\text{c.d.}(H) \subseteq \{1, 2\}$  and let  $K$  be a Hopf subalgebra or quotient Hopf algebra of  $H$ . Then  $\text{c.d.}(K) \subseteq \{1, 2\}$ .*

*Proof.* We only need to show the claim when  $K \subseteq H$  is a Hopf subalgebra. In this case, the statement follows from surjectivity of the restriction functor  $\text{Rep } H \rightarrow \text{Rep } K$ .  $\square$

The lemma has the following immediate consequence:

**Corollary 6.9.** *If  $\text{c.d.}(H) \subseteq \{1, 2\}$ , then the group  $G(H)$  is solvable.*

**Lemma 6.10.** *Suppose  $\text{c.d.}(H), \text{c.d.}(H^*) \subseteq \{1, 2\}$ . Then  $H$  is solvable.*

*Proof.* By induction on the dimension of  $H$ .

Consider the universal grading group  $U$  of the category  $\text{Rep } H$ . Then  $H^* \rightarrow kU$  is a quotient Hopf algebra and therefore  $\text{c.d.}(U) \subseteq \{1, 2\}$ , by Lemma 6.8. This implies that the group  $U$  is solvable.

Suppose first  $H_{\text{ad}} \neq H$ . In view of Lemma 6.8, we also have  $\text{c.d.}(H_{\text{ad}}), \text{c.d.}(H_{\text{ad}}^*) \subseteq \{1, 2\}$ . By the inductive assumption  $H_{\text{ad}}$  is solvable. By [8, Proposition 4.5 (i)],  $H$  is solvable, since  $\text{Rep } H$  is a  $U$ -extension of  $\text{Rep } H_{\text{ad}}$ .

It remains to consider the case where  $H_{\text{ad}} = H$ . As pointed out at the beginning of this subsection, it follows from [2, Theorem 6.4] that in this case  $H$  satisfies condition (i), that is,  $H$  fits into a cocentral abelian exact sequence  $k \rightarrow k^\Gamma \rightarrow H \rightarrow k\Gamma \rightarrow k$ , with  $|\Gamma| > 1$  and  $\Gamma$  abelian.

In particular,  $k^\Gamma \subseteq H^*$  is a nontrivial central Hopf subalgebra, implying that  $H^* \neq H_{\text{ad}}^*$ . The inductive assumption implies, as before, that  $H_{\text{ad}}^*$  and thus also  $H^*$  is solvable. Then  $H$  also is. This finishes the proof of the lemma.  $\square$

**6.2. The quasitriangular case.** We shall assume in this subsection that  $H$  is quasitriangular. Let  $R \in H \otimes H$  be an  $R$ -matrix. We keep the notation in Subsection 2.6.

*Remark 6.11.* Notice that, since the category  $\text{Rep } H$  is braided, then the universal grading group  $U = U(\text{Rep } H)$  is abelian (and in particular, solvable).

The following is the main result of this subsection.

**Theorem 6.12.** *Let  $H$  be a quasitriangular semisimple Hopf algebra such that  $\text{c.d.}(H) \subseteq \{1, 2\}$ . Then  $H$  is solvable.*

*Proof.* If  $\text{c.d.}(H) = \{1\}$ , then  $H$  is commutative and, because it is quasitriangular, isomorphic to the group algebra of an abelian group. Hence we may assume that  $\text{c.d.}(H) = \{1, 2\}$ .

Consider the Hopf subalgebras  $H_+, H_- \subseteq H$ . By Lemma 6.8, we have  $\text{c.d.}(H_+), \text{c.d.}(H_-) \subseteq \{1, 2\}$ . Then  $\text{c.d.}(H_-), \text{c.d.}(H_-^*) \subseteq \{1, 2\}$ , since  $(H_-^*)^{\text{cop}} \simeq H_+$ .

By Lemma 6.10,  $H_-$  is solvable. Therefore the Drinfeld double  $D(H_-)$  and its homomorphic image  $H_R$  are also solvable.

We may thus assume that  $H_R \subsetneq H$ .

Observe that, being a quotient of  $H$ ,  $H_{\text{ad}}$  is also quasitriangular and satisfies  $\text{c.d.}(H_{\text{ad}}) \subseteq \{1, 2\}$ . Hence, by induction, we may also assume that  $H = H_{\text{ad}}$ , and in particular,  $G(H) \cap Z(H) = 1$ . Indeed,  $\text{Rep } H$  is a  $U$ -extension of  $\text{Rep } H_{\text{ad}}$  and the group  $U$  is abelian, as pointed out before.

Therefore  $H$  fits into a cocentral abelian exact sequence  $k \rightarrow k^F \rightarrow H \rightarrow k\Gamma \rightarrow k$ , where  $1 \neq \Gamma$  is elementary abelian of exponent 2.

In view of Lemma 6.6, it will be enough to show that the group  $F$  is solvable.

We have  $\widehat{\Gamma} \subseteq G(H^*) \cap Z(H^*)$ . By [29, Proposition 3],  $f_{R_{21}}(G(H^*) \cap Z(H^*)) \subseteq G(H) \cap Z(H)$ . Hence we may assume that  $f_{R_{21}}|_{\widehat{\Gamma}} = 1$  and similarly  $f_R|_{\widehat{\Gamma}} = 1$ . Thus  $f_R, f_{R_{21}}$  factorize through the quotient  $H^*/H^*(k\widehat{\Gamma})^+ \simeq kF$ .

Therefore  $H_+ = f_R(H^*)$  and  $H_- = f_{R_{21}}(H^*)$  are cocommutative. (Then they are also commutative, since  $H_+ \simeq H_-^{\text{cop}}$ .) In particular,  $H_R = H_+H_-$  is cocommutative. Hence  $\Phi_R(H^*) \subseteq H_R \subseteq kG(H)$ .

By [22, Theorem 4.11],  $K = \Phi_R(H^*)$  is a commutative (and cocommutative) normal Hopf subalgebra, which is necessarily solvable, since  $H_R$  is. In addition,  $\Phi_R(H^*) \simeq kT$ , where  $T \subseteq G(H)$  is an abelian subgroup ([22, Example 2.1]), and there is an exact sequence of Hopf algebras

$$k \rightarrow kT \rightarrow H \xrightarrow{\pi} \overline{H} \rightarrow k,$$

where  $\overline{H}$  is a certain (canonical) triangular Hopf algebra.

Since  $\overline{H}$  is triangular, then  $\overline{H} \simeq (kL)^J$ , is a twisting of the group algebra of some finite group  $L$ . Because  $\text{c.d.}(L) = \text{c.d.}(\overline{H}) \subseteq \{1, 2\}$ ,  $L$  must be solvable. Hence  $\overline{H}$  is solvable, since  $\text{Rep } \overline{H} \simeq \text{Rep } L$ .

The map  $\pi : H \rightarrow \overline{H}$  induces, by restriction to the Hopf subalgebra  $k^F \subseteq H$ , an exact sequence

$$k \rightarrow kT \cap k^F \rightarrow k^F \xrightarrow{\pi|_{k^F}} \pi(k^F) \rightarrow k.$$

We have  $kT \cap k^F = k^{\overline{F}}$  and  $\pi(k^F) = k^S$ , where  $\overline{F}$  and  $S$  are a quotient and a subgroup of  $F$ , respectively, in such a way that the exact sequence above corresponds to an exact sequence of groups

$$1 \rightarrow S \rightarrow F \rightarrow \overline{F} \rightarrow 1.$$

Now,  $\overline{F}$  is abelian, because  $k^{\overline{F}} = kT \cap k^F$  is cocommutative, and  $S$  is solvable, because  $k^S$  is a Hopf subalgebra of  $\overline{H}$ . Therefore  $F$  is solvable. This implies that  $H$  is solvable and finishes the proof of the theorem.  $\square$

## 7. ODD DIMENSIONAL FUSION CATEGORIES

Along this section,  $p$  will be a prime number. Let  $\mathcal{C}$  be a fusion category over  $k$ . Recall that the set of irreducible degrees of  $\mathcal{C}$  was defined as

$$\text{c.d.}(\mathcal{C}) = \{\text{FPdim } x \mid x \in \text{Irr } \mathcal{C}\}.$$

The fusion categories that we shall consider in this section are all *integral*, that is, the Frobenius-Perron dimensions of objects of  $\mathcal{C}$  are (natural) integers. By [7, Theorem 8.33],  $\mathcal{C}$  is isomorphic to the category of representations of some finite dimensional semisimple quasi-Hopf algebra.

### 7.1. Odd dimensional weakly group-theoretical fusion categories.

The following result is a consequence of the Feit-Thompson theorem [9].

**Proposition 7.1.** *Let  $\mathcal{C}$  be a weakly group-theoretical fusion category and assume that  $\text{FPdim } \mathcal{C}$  is an odd integer. Then  $\mathcal{C}$  is solvable.*

Note that since  $\text{FPdim } \mathcal{C}$  is an odd integer, the fusion category  $\mathcal{C}$  is integral. See [5, Corollary 2.22].

*Proof.* By definition,  $\mathcal{C}$  is Morita equivalent to a nilpotent fusion category. Then, by [8, Proposition 4.5 (i)], it will be enough to show that a nilpotent fusion category of odd Frobenius-Perron dimension is solvable. So, assume that  $\mathcal{C}$  is nilpotent, so that  $\mathcal{C}$  is a  $G$ -extension of a fusion subcategory  $\tilde{\mathcal{C}}$ , with  $|G| > 1$ . In particular,  $\text{FPdim } \mathcal{C} = |G| \text{FPdim } \tilde{\mathcal{C}}$ . Hence the order of  $G$  and  $\text{FPdim } \tilde{\mathcal{C}}$  are odd and  $\text{FPdim } \tilde{\mathcal{C}} < \text{FPdim } \mathcal{C}$ . The theorem follows by induction, since by the Feit-Thompson theorem,  $G$  is solvable. See [8, Proposition 4.5 (i)].  $\square$

**7.2. Braided fusion categories.** We shall need the following lemma whose proof is contained in the proof of [8, Proposition 6.2 (i)]. We include a sketch of the argument for the sake of completeness.

**Lemma 7.2.** *Let  $\mathcal{C}$  be a fusion category and let  $G$  be a finite group acting on  $\mathcal{C}$  by tensor autoequivalences. Assume  $\text{c.d.}(\mathcal{C}^G) \subseteq \{p^m : m \geq 0\}$ , where  $p$  is a prime number. Then  $\text{c.d.}(\mathcal{C}) \subseteq \{p^m : m \geq 0\}$ .*

*Proof.* Regard  $\mathcal{C}$  as an indecomposable module category over itself via tensor product, and similarly for  $\mathcal{C}^G$ . Let  $Y$  be a simple object of  $\mathcal{C}$ . Since the forgetful functor  $F : \mathcal{C}^G \rightarrow \mathcal{C}$  is surjective,  $Y$  is a simple constituent of  $F(X)$ , for some simple object  $X$  of  $\mathcal{C}^G$ .

Since  $F$  is a tensor functor, we have  $\text{FPdim } X = \text{FPdim } F(X)$ . By Formula (7) in [8, Proof of Proposition 6.2],

$$(7.1) \quad \text{FPdim}(X) = \deg(\pi)[G : G_Y] \text{FPdim } Y,$$

where  $G_Y \subseteq G$  is the stabilizer of  $Y$  and  $\pi$  is an irreducible representation of  $G_Y$  associated to  $X$ . Therefore  $\text{FPdim } Y$  divides  $\text{FPdim } X$ .

The assumption on  $\mathcal{C}^G$  implies that  $\text{FPdim } X$  is a power of  $p$ . Then so is  $\text{FPdim } Y$ . This proves the lemma.  $\square$

**Theorem 7.3.** *Let  $\mathcal{C}$  be a braided fusion category such that  $\text{c.d.}(\mathcal{C}) \subseteq \{p^m : m \geq 0\}$ , where  $p$  is a prime number. Assume that  $\text{FPdim } \mathcal{C}$  is odd. Then  $\mathcal{C}$  is solvable.*

*Proof.* By induction on  $\text{FPdim } \mathcal{C}$ . (Notice that the Frobenius-Perron dimension of a fusion subcategory of  $\mathcal{C}$  divides the dimension of  $\mathcal{C}$  [7, Proposition 8.15], and the same is true for the Frobenius-Perron dimension of a fusion category  $\mathcal{D}$  such that there exists a surjective tensor functor  $\mathcal{C} \rightarrow \mathcal{D}$  [7, Corollary 8.11]. Thus these fusion categories are odd-dimensional as well.) If  $\text{c.d.}(\mathcal{C}) = \{1\}$ , then  $\mathcal{C}$  is pointed. Then  $\mathcal{C} \simeq \mathcal{C}(G, \omega)$  for some abelian

group  $G$  and some 3-cocycle  $\omega$  on  $G$ . Then  $\mathcal{C}$  is solvable, by [8, Proposition 4.5 (ii)].

Suppose next that  $\mathcal{C}$  is not pointed. Then all non-invertible objects in  $\mathcal{C}$  have Frobenius-Perron dimension  $p^m$ , for some  $m \geq 1$ . Consider the group  $G(\mathcal{C})$  of invertible objects of  $\mathcal{C}$ . Then  $G(\mathcal{C})$  is abelian and  $G(\mathcal{C}) \neq 1$ , as follows by taking Frobenius-Perron dimensions in a decomposition of the tensor product  $X \otimes X^*$ , for some simple non-invertible object  $X$ .

Let us regard  $\mathcal{C}$  as a premodular fusion category with respect to its canonical spherical structure (as  $\text{FPdim } \mathcal{C}$  is an integer). Then  $\mathcal{C}$  is modularizable, in view of [4, Lemma 7.2].

Let  $\tilde{\mathcal{C}}$  be its modularization, which is a modular category over  $k$ . Then  $\mathcal{C}$  is an equivariantization  $\mathcal{C} \simeq \tilde{\mathcal{C}}^G$  with respect to the action of a certain group  $G$  on  $\tilde{\mathcal{C}}$  [3]. (Indeed, the modularization functor  $\mathcal{C} \rightarrow \tilde{\mathcal{C}}$  gives rise to an exact sequence of fusion categories  $\text{Rep } G \rightarrow \mathcal{C} \rightarrow \tilde{\mathcal{C}}$ , which comes from an equivariantization; see [4, Example 5.33].)

By construction of  $G$ , the category  $\text{Rep } G$  is the (tannakian) fusion subcategory of transparent objects in  $\mathcal{C}$ . Therefore there is an embedding of braided fusion categories  $\text{Rep } G \subseteq \mathcal{C}$ . In particular, the order of  $G$  is odd, implying that  $G$  is solvable.

By Lemma 7.2,  $\text{c.d.}(\tilde{\mathcal{C}}) \subseteq \{p^m : m \geq 0\}$ . Then, by induction, and since an equivariantization of a solvable fusion category under the action of a solvable group is again solvable, we may and shall assume in what follows that  $\mathcal{C} = \tilde{\mathcal{C}}$  is modular.

It is shown in [11, Theorem 6.2] that the universal grading group  $U(\mathcal{C})$  is (abelian and) isomorphic to the group  $\widehat{G(\mathcal{C})}$  of characters of  $G(\mathcal{C})$ . In particular,  $U(\mathcal{C}) \neq 1$ . On the other hand,  $\mathcal{C}$  is a  $U(\mathcal{C})$ -extension of its fusion subcategory  $\mathcal{C}_{\text{ad}}$ . Since also  $\text{c.d.}(\mathcal{C}_{\text{ad}}) \subseteq \{p^m : m \geq 0\}$ , then  $\mathcal{C}_{\text{ad}}$  is solvable, by induction. Therefore  $\mathcal{C}$  is solvable, as claimed.  $\square$

## REFERENCES

- [1] B. BAKALOV and A. KIRILLOV JR. *Lectures on Tensor categories and modular functors*, University Lecture Series **21**, Am. Math. Soc., Providence, 2001.
- [2] J. BICHON and S. NATALE, *Hopf algebra deformations of binary polyhedral groups*, Transf. Groups **16** (2) 339–374 (2011).
- [3] A. BRUGUIÈRES, *Catégories prémodulaires, modularisations et invariants des variétés de dimension 3*, Math. Ann. **316**, 215–236 (2000).
- [4] A. BRUGUIÈRES and S. NATALE, *Exact sequences of tensor categories*, Int. Math. Res. Not. doi:10.1093/imrn/rnq294, 1–62 (2011).
- [5] V. DRINFELD, S. GELAKI, D. NIKSHYCH and V. OSTRIK, *On braided fusion categories I*, Sel. Math. New Ser. **16**, 1–119 (2010).
- [6] P. ETINGOF and S. GELAKI, *The classification of finite dimensional triangular Hopf algebras over an algebraically closed field of characteristic 0*, Mosc. Math. J. **3**, 37–43 (2003).
- [7] P. ETINGOF, D. NIKSHYCH and V. OSTRIK, *On fusion categories*, Annals of Mathematics **162**, 581–642 (2005).



- [8] P. ETINGOF, D. NIKSHYCH and V. OSTRIK, *Weakly group-theoretical and solvable fusion categories*, Adv. Math. **226**, 176–205 (2011).
- [9] W. FEIT and J. THOMPSON, *Solvability of groups of odd order*, Pacific J. Math. **13**, 775–1029 (1963).
- [10] S. GELAKI and D. NAIDU, *Some properties of group-theoretical categories*, J. Algebra **322**, 2631–2641 (2009).
- [11] S. GELAKI and D. NIKSHYCH, *Nilpotent fusion categories*, Adv. Math. **217**, 1053–1071 (2008).
- [12] I. ISAACS, *Character theory of finite groups*, Pure and Applied Mathematics **69**, Academic Press, New York, 1976.
- [13] M. IZUMI and H. KOSAKI *Kac algebras arising from composition of subfactors: general theory and classification*, Mem. Amer. Math. Soc. **158**, 750, (2007).
- [14] Y. KASHINA, G. MASON and S. MONTGOMERY, *Computing the Frobenius-Schur indicator for abelian extensions of Hopf algebras*, J. Algebra **251**, 888–913 (2002).
- [15] A. MASUOKA, *Semisimple Hopf algebras of dimension 6, 8*, Israel J. Math. **92**, 361–373 (1995).
- [16] A. MASUOKA, *Extensions of Hopf algebras*, Trabajos de Matemática **41/99**, Universidad Nacional de Córdoba, 1999.
- [17] A. MASUOKA, *Hopf algebra extensions and cohomology*, Math. Sci. Res. Inst. Publ. **43**, 167–209 (2002).
- [18] S. MONTGOMERY and S. WHITERSPOON, *Irreducible representations of crossed products*, J. Pure Appl. Algebra **129**, 315–326 (1998).
- [19] S. NATALE, *On semisimple Hopf algebras of dimension  $pq^2$* , J. Algebra **221**, 242–278 (1999).
- [20] S. NATALE, *On group-theoretical Hopf algebras and exact factorizations of finite groups*, J. Algebra **270**, 199–211 (2003).
- [21] S. NATALE, *Frobenius-Schur indicators for a class of fusion categories*, Pacific J. Math. **221**, 353–378 (2005).
- [22] S. NATALE, *R-matrices and Hopf algebra quotients*, Int. Math. Res. Not. **2006**, 1–18 (2006).
- [23] S. NATALE, *On the exponent of tensor categories coming from finite groups*, Israel J. Math. **162**, 253–273 (2007).
- [24] S. NATALE, *Semisolvability of Semisimple Hopf Algebras of Low Dimension*, Mem. Amer. Math. Soc. **186**, 874, (2007).
- [25] S. NATALE, *Semisimple Hopf algebras and their representations*, to appear in Publ. Mat. Uruguay (2010).
- [26] W. NICHOLS and M. RICHMOND, *The Grothendieck group of a Hopf algebra*, J. Pure Appl. Algebra **106**, 297–306 (1996).
- [27] W. NICHOLS and M. ZOELLER, *A Hopf algebra freeness Theorem*, Amer. J. Math. **111**, 381–385 (1989).
- [28] V. OSTRIK, *Module categories over the Drinfeld double of a finite group*, Int. Math. Res. Not. **2003**, 1507–1520 (2003).
- [29] D. RADFORD, *On the antipode of a quasitriangular Hopf algebra*, J. Algebra **151**, 1–11 (1992).
- [30] D. RADFORD, *Minimal quasitriangular Hopf algebras*, J. Algebra **157**, 285–315 (1993).
- [31] N. RESHETIKHIN and M. SEMENOV-TIAN-SHANSKY *Quantum R-matrices and factorization problems*, J. Geom. Phys. **5**, 533–550 (1988).
- [32] D. ROBINSON, *A course in the theory of groups*, Graduate Texts Math. **80**, Springer-Verlag, Berlin, 1982.
- [33] H.-J. SCHNEIDER, *Some properties of factorizable Hopf algebras*, Proc. Amer. Math. Soc. **29**, 1891–1898 (2001).
- [34] G. SEITZ, *Finite groups having only one irreducible representation of degree greater than one*, Proc. Amer. Math. Soc. **19**, 459–461 (1968).

- [35] J. SIEHLER, *Near-group categories*, Algebr. Geom. Topol. **3**, 719–775 (2003).
- [36] Y. SOMMERHÄUSER, *Yetter-Drinfel'd Hopf algebras over groups of prime order*, Lectures Notes in Math. **1789**, Springer-Verlag (2002).
- [37] D. TAMBARA and S. YAMAGAMI, *Tensor categories with fusion rules of self-duality for finite abelian groups*, J. Algebra **209**, 692–707 (1998).
- [38] H. WIELANDT, *Über Produkte von nilpotenten Gruppen*, Illinois J. Math. **2**, 611–618 (1958).
- [39] S. ZHU, *On finite dimensional semisimple Hopf algebras*, Commun. Algebra **21**, 3871–3885 (1993).

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